Exact controllability for the three–dimensional Navier–Stokes equations with the Navier slip boundary conditions

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Abstract

In this paper we establish the local exact internal controllability of steady state solutions for the Navier–Stokes equations in three–dimensional bounded domains, with the Navier slip boundary conditions. The proof is based on a Carleman–type estimate for the backward Stokes equations with the same boundary conditions, which is also established here.

Key words: Navier–Stokes equations, controllability, observability inequality, Carleman estimate.

Mathematics Subject Classifications 2000: 35Q30, 76D05, 76D55, 35B37, 93B05, 93C20, 93B07.

1 Introduction

In this paper we deal with the internal controllability of the steady state solutions of the Navier–Stokes equations in three–dimensional bounded domains and with the Navier–slip boundary conditions. We shall show that the steady state (stationary) solutions of the Navier–Stokes equations with the Navier slip boundary conditions are locally exactly controllable provided that they are sufficiently smooth. More specifically, it will be proved that it is possible to reach such a steady state solution in a given finite time if one starts from initial data which are "close" to this solution and one acts by suitable locally distributed external forces (as internal controls).

In two dimensions a similar result was obtained by O.Yu. Imanuvilov in [8]. The analogous controllability result for the Navier–Stokes equations with the no–slip boundary condition was established (in two and three dimensions) by Imanuvilov, too, in [9] and [10]. (See also [2] and [3] for a slightly different approach.)

We shall reduce the local controllability for the Navier–Stokes equations to the global controllability for the linearized Navier–Stokes equations by means of an infinite–dimensional version of the local inversion theorem. (The same approach was used in [9] and [10].) To obtain the global controllability for the linearized system, we need an observability inequality for the backward adjoint system. One can prove such an inequality by using a Carleman–type estimate for the backward Stokes equations with the Navier boundary conditions. So, much of the substance of this paper will consist in deriving such an estimate.
2 Functional framework and main result

Let $\Omega$ be a bounded multi-connected open set in $\mathbb{R}^3$ whose boundary $\partial \Omega$ is a finite union of mutually disjoint two-dimensional connected manifolds of class $C^2$, and let $T > 0$ be a fixed time. It is clear that $\Omega$ can be made simply connected with a finite number of smooth cuts, that is, there exist $p$ mutually disjoint two-dimensional manifolds $\Gamma_1, ..., \Gamma_p$ of class $C^2$ which are not tangent to $\partial \Omega$ such that $\Omega \setminus (\bigcup_{i=1}^p \Gamma_i)$ is simply connected. We set $Q = \Omega \times (0,T)$. We consider an open subset $\omega$ of $\Omega$, too. The controlled Navier–Stokes equations (with the Navier slip boundary and initial conditions) we deal with are the following:

\[
\begin{align*}
\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla)y + \nabla p &= f + \chi_\omega u & \text{in } Q, \\
\text{div } y &= 0 & \text{in } Q, \\
y \cdot N &= 0 & \text{on } \Sigma = \partial \Omega \times (0,T), \\
\sum_{i,j=1}^3 \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) N_i T_j &= 0 & \text{on } \Sigma, \\
& \text{for all tangential vector fields } T = (T_1, T_2, T_3) & \text{on } \partial \Omega \\
& (i.e., T \cdot N = 0 \text{ on } \partial \Omega), \\
y(x, 0) &= y_0 & \text{in } \Omega.
\end{align*}
\]

Here $y = (y_1, y_2, y_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the velocity vector field, $p : \Omega \times [0,T] \to \mathbb{R}$ is the (scalar) pressure and $u = (u_1, u_2, u_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the control parameter. We denote the variables of the functions (fields) $y, p, u$ by $x = (x_1, x_2, x_3)$ (belonging to $\Omega$) and $t$ (in $[0,T]$). Further, $\nu$ is the kinematic viscosity coefficient, which (without loss of generality) will be supposed to be 1, $f = (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ is the known density of the external forces, $\chi_\omega$ is the characteristic function of $\omega$, and $y_0 : \Omega \to \mathbb{R}^3$ is the given initial velocity field. Finally, $N$ is the unit outer normal to $\partial \Omega$.

In the second boundary condition in (2.1), the first order factors $\partial y_i/\partial x_j + \partial y_j/\partial x_i$ come from the stress tensor $\sigma$ of components $\sigma_{ij} = \mu(\partial y_i/\partial x_j + \partial y_j/\partial x_i) - p\delta_{ij}$. (Here the pressure $p$ dissapears by the multiplication of $\sigma_{ij}$ by $N_i T_j$.) The Navier slip boundary conditions in (2.1) can be viewed as a kind of intermediate condition between the Dirichlet–type boundary condition $y = 0$ (the no–slip boundary condition) and the Neumann–type boundary conditions arising in the free boundary fluid flow. One could imagine a
physical situation in which the boundary conditions in (2.1) occur as follows: the outer region is filled by an inviscid fluid having much greater density than the fluid filling $\Omega$ (see [7]).

In the case when $\Omega$ is not simply connected (but it is multi-connected), to assure the well-posedness of problem (2.1), we need to impose some supplementary conditions for $y$ on the cuts $\Gamma_i$:

\[(2.2) \quad \int_{\Gamma_i} y \cdot N d\sigma = 0, \quad i = 1, ..., p,\]

where $N$ is the unit outer normal to $\Gamma_i$'s. (See Lemmas 1.1–1.4 in Appendix I of [17] for an equivalent form of (2.2).) Of course, if $\Omega$ is simply connected, condition (2.2) is no longer necessary.

To formulate our results we need several function spaces. For each positive integer $m$ and $p > 1$, or $p = +\infty$, we denote (as usual) by $W^{m,p}(\Omega)$ the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to $m$ are also in $L^p(\Omega)$. When $p = 2$, instead of $W^{m,2}(\Omega)$ we shall write $H^m(\Omega)$. The space $H^{2,1}(Q)$ contains those functions in $L^2(Q)$ whose first and second order weak derivatives with respect to the space variables $x_1, x_2, x_3$ and first order weak derivative with respect to $t$ belong to $L^2(Q)$, too. Our subsequent considerations also require some fractional order Sobolev spaces $H^s(\Omega)$ and trace spaces $H^s(\partial\Omega)$ with $s > 0$. (We refer the reader to [12] for definitions and more information concerning these spaces.) Besides, we need the space $L^2(0, T; H^1(\Omega))$ of all (equivalence classes of) measurable functions from $(0, T)$ to $H^1(\Omega)$ having the square of their $H^1$ norm integrable over $(0, T)$. The space $L^2(0, T; H^1(\partial\Omega))$ is analogously defined.

Because $y, \nabla p, u$ and $f$ in the Navier–Stokes equations (we are concerned with) are actually vector fields, we can view them as belonging to some product function spaces: $(L^2(\Omega))^3$, $(H^1(\Omega))^3$, $(L^2(Q))^3$, $(H^{2,1}(Q))^3$, $L^2(0, T; (H^1(\Omega))^3)$, etc, all of them endowed with the product norms. Finally, we denote by $((H^1_0(\Omega))^3)'$ the dual of the space $(H^1_0(\Omega))^3$ (of the vector fields in $(H^1(\Omega))^3$ that are equal to zero on the boundary). The norms of all the considered spaces are denoted in the following way: $| \cdot |_{L^2(\Omega)}, | \cdot |_{(L^2(\Omega))^3}, | \cdot |_{(L^2(\Omega))^3}$, $| \cdot |_{(H^{2,1}(Q))^3}$, etc.

The Navier–Stokes equations can be expressed as an evolution equation in the space $H$ of all weakly divergence-free vector fields in $(L^2(\Omega))^3$ which are tangential to the boundary in a weak sense, endowed with the $(L^2(\Omega))^3$ norm:
\[ H = \{ y \in (L^2(\Omega))^3 : \text{div} \, y = 0 \text{ in } \Omega \text{ and } y \cdot N = 0 \text{ on } \partial \Omega \} \].

In fact, here we need the following \( H^1 \) variant of this space:

\[ V = \{ y \in (H^1(\Omega))^3 : \text{div} \, y = 0 \text{ in } \Omega \text{ and } y \cdot N = 0 \text{ on } \partial \Omega \} \],

endowed with the \((H^1(\Omega))^3\) norm. One can show that all fields \( y \) in \( V \) satisfying (2.2) also satisfy the following Poincaré–type inequality:

\[
\frac{1}{2} \int_\Omega \sum_{i,j=1}^3 \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right)^2 \, dx = \int_\Omega |\text{curl} \, y|^2 \, dx \geq c \int_\Omega |y|^2 \, dx,
\]

where \( c \) is a positive constant depending only on \( \Omega \). (See Lemma 1.6 in Appendix I of [17].)

Now we consider a steady state (equilibrium) solution \((y_e, p_e)\) of (2.1) and (2.2). So it satisfies

\[
\begin{align*}
-\nabla y_e + (y_e \cdot \nabla)y_e + \nabla p_e &= f \quad \text{in } \Omega, \\
\text{div} \, y_e &= 0 \quad \text{in } \Omega, \\
y_e \cdot N &= 0 \quad \text{on } \partial \Omega, \\
\sum_{i,j=1}^3 \left( \frac{\partial y_{ei}}{\partial x_j} + \frac{\partial y_{ej}}{\partial x_i} \right) N_i T_j &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

for all tangential vector fields \( T = (T_1, T_2, T_3) \) on \( \partial \Omega \).

The main result of this paper, Theorem 2.1 below, asserts that \( y_e \) is locally exactly controllable if it is smooth enough; that is, for \( y_0 \) sufficiently "close" to \( y_e \) there exist \( u, y \) and \( p \) that satisfy equations (2.1) and (2.2).

**Theorem 2.1.** Let \( \Omega \) be an open, bounded, and multi–connected subset of \( \mathbb{R}^3 \) whose boundary \( \partial \Omega \) is a finite union of mutually disjoint two–dimensional compact connected manifolds of class \( C^2 \), and let \( \omega \) be an open subset of \( \Omega \). Let \( f \in (L^2(\Omega))^3 \) and let \((y_e, p_e) \in (W^{2,4}(\Omega))^3 \times H^1(\Omega) \) be a steady state solution of (2.1) and (2.2) (that is, it satisfies (2.3)). Then there is \( \eta > 0 \) such that for any \( y_0 \in V \) satisfying

\[
|y_0 - y_e|_{(H^1(\Omega))^3} \leq \eta
\]

there exists \((u, y, p) \in (L^2(Q))^3 \times (H^{2,1}(Q))^3 \times L^2(0, T; H^1(\Omega)) \) which satisfies (2.1), (2.2), and

\[
y(x, T) = y_e(x) \quad \text{a.e. } x \in \Omega.
\]
The statement of Theorem 2.1 still remains valid if, as in [10], instead of a stationary solution \((y_e, p_e)\) as target, we take a solution \((\tilde{y}, \tilde{p})\) of (2.1) and (2.2) satisfying an analogous regularity condition: \((\tilde{y}, \tilde{p}) \in H^1(0, T; (W^{2,4}(\Omega))^3) \times L^2(0, T; H^1(\Omega))\). In this case, the state \(\tilde{y}(\cdot, T)\) is locally exactly controllable. This means that there is \(\eta > 0\) such that for any \(y_0 \in V\) satisfying

\[|y_0 - \tilde{y}(\cdot, 0)|_{(H^1(\Omega))^3} \leq \eta\]

there exist \(u, y\) and \(p\) as in Theorem 2.1, which satisfy (2.1), (2.2), and

\[y(x, T) = \tilde{y}(x, T) \text{ a.e. } x \in \Omega.\]

The proof of this result follows exactly the same lines as that of Theorem 2.1.

We also note that, by the Sobolev imbedding theorem, the condition \(y_e \in (W^{2,4}(\Omega))^3\) is satisfied if \(y_e \in (H^3(\Omega))^3\).

Now let us briefly describe the proof of Theorem 2.1, which will be developed in the following three sections. As we have already mentioned in Introduction, the local controllability for the Navier–Stokes equations will be reduced to the global controllability for their linearized version by means of an infinite–dimensional variant of the local inversion theorem. More specifically, taking the difference of equations (2.1) and (2.3), the controllability of a steady state solution for the Navier–Stokes equations is immediately reduced to that of the null solution for slightly modified Navier–Stokes equations (with null external forces). Then we shall reformulate the local null controllability for the (modified) Navier–Stokes equations (in fact the existence of \(u, y\) and \(p\) asserted by Theorem 2.1) as a local right invertibility property for a certain nonlinear map, suitably chosen. The condition that the derivative of this map should be an epimorphism (that is, the sufficient condition for the map to be a local epimorphism) is equivalent to the global null controllability property for the linearization of the Navier–Stokes equations around the state \(y_e\). The global controllability problem will be then viewed as the ”limit” of a family of optimal control problems for the linearized Navier–Stokes equations when a certain penalization parameter (multiplying the \(L^2\) norm of the final state in the functional we have to minimize) tends to infinity. We shall show that the solutions of the auxiliary optimal control problems converge to a solution of the global controllability problem for the linearized equations. The needed estimates for the passage to the limit require an observability
inequality for the backward adjoint equations. This inequality is derived from a Carleman–type estimate for the backward Stokes equations (with the Navier boundary conditions). We shall reverse this description by beginning the proof with the establishing of the Carleman estimate.

3 Carleman inequality for the backward Stokes equations with the Navier boundary conditions

Let $\Omega$ and $\omega$ be open sets as in the statement of Theorem 2.1. The backward Stokes equations with the Navier boundary conditions are the following:

\begin{equation}
\frac{\partial z}{\partial t} + \Delta z + \nabla q = g \quad \text{in } Q=\Omega \times (0, T),
\end{equation}
\begin{equation}
\text{div } z = 0 \quad \text{in } Q,
\end{equation}
\begin{equation}
z \cdot N = 0 \quad \text{on } \Sigma=\partial \Omega \times (0, T),
\end{equation}
\begin{equation}
\sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) N_i T_j = 0 \quad \text{on } \Sigma,
\end{equation}
\begin{equation}
\text{for all tangential vector fields } T=(T_1, T_2, T_3) \text{ on } \partial \Omega.
\end{equation}

Essentially, the Carleman inequality for the solutions of (3.1) is an \textit{a priori} estimate which contains only the restriction of solution on $Q_\omega = \omega \times (0, T)$ in the right–hand side, instead of the solution taken on the entire domain $Q$. But to be possible such an estimate, it is necessary to multiply the solution by some suitable weight functions.

Let us introduce the auxiliary functions needed to express the Carleman inequality. Let $\omega_0$ be an arbitrary but fixed open subset of $\omega$ such that $\omega_0 \subset \subset \omega$. Since $\Omega$ is bounded and connected, one can construct a function $\psi \in C^2(\overline{\Omega})$ which satisfies:

\begin{equation}
\psi > 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial \Omega \quad \text{and } \quad |\nabla \psi| > 0 \text{ in } \overline{\Omega} \setminus \omega_0.
\end{equation}

(We refer the reader to [10] for details.) Now we set
\begin{equation}
\varphi(x,t) = \frac{e^{\lambda \psi(x)}}{(t(T-t))^8} \quad \text{and} \quad \alpha(x,t) = \frac{e^{\lambda \psi(x)}}{(t(T-t))^8} - \frac{e^{2\lambda \psi|_{C(\overline{\Omega})}}}{(t(T-t))^8},
\end{equation}
where $\lambda > 0$. We denote by $\hat{\varphi}(t)$ and $\hat{\alpha}(t)$ the values taken by $\varphi$ and $\alpha$ on the boundary $\partial \Omega$ (where $\psi = 0$):

$$\hat{\varphi}(t) = \frac{1}{(t(T-t))^8} \quad \text{and} \quad \hat{\alpha}(t) = \frac{1 - e^{2\lambda|\psi|_{C(\mathbb{R})}}}{(t(T-t))^8}. $$

We shall also use the functions $\varphi$ and $\alpha$ defined as

$$\varphi(x, t) = e^{-\lambda\psi(x)}(t(T-t))^8 \quad \text{and} \quad \alpha(x, t) = e^{-\lambda\psi(x)} - e^{2\lambda|\psi|_{C(\mathbb{R})}}(t(T-t))^8. $$

We set $Q_{\omega_0} = \omega_0 \times (0, T)$ and (as it is already written before) $Q_\omega = \omega \times (0, T)$.

Now we are prepared to present the Carleman inequality for (3.1).

**Theorem 3.1.** Let $\Omega$ be an open, bounded, and multi–connected subset of $\mathbb{R}^3$ whose boundary $\partial \Omega$ is a finite union of mutually disjoint two–dimensional compact connected manifolds of class $C^2$, and let $\omega, \omega_0$ and $\omega_1$ be open subsets of $\Omega$ such that $\omega_0 \subset \subset \omega \subset \subset \omega_1$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ one can find $s_0(\lambda) > 0$ and $c(\lambda) > 0$ that for $s > s_0(\lambda)$ the following inequality holds:

$$\int_Q e^{2s\alpha \left( \frac{1}{s\varphi} \left( \frac{\partial z}{\partial t} \right)^2 + \sum_{i,j=1}^3 \frac{1}{(\partial x_i \partial x_j)^2} \right) + s\varphi |\nabla z|^2 + s^3 \varphi^3 |z|^2} \, dx \, dt$$

$$\leq c(\lambda) \left( \int_{Q_{\omega_0}} e^{2s\alpha g^2} \, dx \, dt + \int_{Q_{\omega_1}} e^{2s\alpha s^3} \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha s^3} \varphi^3 |g|^2 \, dx \, dt + \int_Q e^{2s\alpha s^3} \varphi^2 |\text{div} \, g|^2 \, dx \, dt \right)$$

for all $g \in L^2(0, T; (H^1(\Omega))^3)$ and all corresponding solutions $(z, q) \in (H^{2,1}(Q))^3 \times L^2(0, T; H^1(\Omega))$ of system (3.1).

For the Stokes equations with the no–slip boundary condition, estimate (3.3) was established by Imanuvilov in [10]. In the main lines, the proof of (3.3) for the Stokes equations with the Navier boundary conditions (we shall present in the following) is similar to those given in [9] or [10], but the different boundary conditions here require a different treatment of the surface integrals arising in the integrations by parts we have to do. (Actually, the Navier slip boundary conditions are much closer to the so–called perfectly conductive
wall conditions occurring for the magnetic field in the magnetohydrodynamic equations. We refer the reader to [4] for the treatment of this case.) For this reason, and to have an independent presentation, we shall develop the proof in almost all its details.

Proof of Theorem 3.1. The proof essentially contains three stages: the estimate of \( z \) and its first and second order derivatives in terms of its restriction on \( Q_\omega \), the gradient of the pressure on \( Q \), and \( g \); the estimate for the pressure in terms of its restriction on \( Q_\omega \), \( z \) on the entire \( Q \), \( g \), and \( \text{div} \ g \). The coupling of the two estimates finishes the proof.

1. The estimate of \( z \). The first needed inequality is contained in the following statement.

**Lemma 3.1.** Let \( \Omega \) be an open, bounded, and connected subset of \( \mathbb{R}^3 \) having the boundary \( \partial \Omega \) of class \( C^2 \), and let \( \omega \) be an open subset of \( \Omega \). Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda > \lambda_0 \) one can find \( s_0(\lambda) > 0 \) and \( c(\lambda) > 0 \) that for \( s > s_0(\lambda) \) we have

\[
\int_{Q} e^{2s \alpha} \left( \frac{1}{s \varphi} \left( \frac{\partial z}{\partial t} \right)^2 + \sum_{i,j=1}^{3} \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + s \varphi |\nabla z|^2 + s^3 \varphi^3 |z|^2 \right) \, dx \, dt \\
\leq c(\lambda) \left( \int_{Q_\omega} e^{2s \alpha} s^3 \varphi^3 |z|^2 \, dx \, dt + \int_{Q} e^{2s \alpha} |\nabla q|^2 \, dx \, dt + \int_{Q} e^{2s \alpha} |g|^2 \, dx \, dt \right)
\]

for all \( g \in (L^2(Q))^3 \) and all corresponding solutions \( (z, q) \in (H^{2,1}(Q))^3 \times L^2(0, T; H^1(\Omega)) \) of system (3.1).

**Proof.** To obtain (3.4), we shall treat system (3.1) in the manner in which the backward dynamo equations in domains with perfectly conductive boundary were treated in [4] to derive an analogous Carleman estimate. In fact, in the two situations we have similar first order boundary conditions. (For more comments on the reasons behind the choice of the ways used in the proof, we refer the reader to [4], Theorem 3.2.)

First we shall prove a variant of (3.4) having only \( z \) and \( \nabla z \) in the left–hand side. (This part contains the key point where the integral over \( Q_\omega \) in the right–hand side will appear.) To this end, we set \( w = e^{s \alpha} z \). Changing \( z \) by \( w \) in (3.1), we have
\[ \frac{\partial w}{\partial t} + \Delta w + s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 w - 2s \lambda \varphi (\nabla \psi \cdot \nabla) w - s \lambda \Delta \psi w - s \frac{\partial \alpha}{\partial t} w = e^{s\alpha} g - e^{s\alpha} \nabla q \quad \text{in } Q, \]
\[ \text{div } w = s \lambda \varphi (\nabla \psi \cdot w) \quad \text{in } Q, \]
\[ w \cdot N = 0 \quad \text{on } \Sigma, \]
\[ \sum_{i,j=1}^{3} \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) N_i T_j = -s \lambda \varphi |\nabla \psi|(w \cdot T) \quad \text{on } \Sigma, \]
\[ w(\cdot, 0) = w(\cdot, T) = 0 \quad \text{on } \Omega. \]

If we denote
\[ P(x,t)w = -\Delta w - s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 w - s \lambda^2 \varphi |\nabla \psi|^2 w + s \lambda \varphi \Delta \psi w + s \frac{\partial \alpha}{\partial t} w, \]
\[ R(x,t)w = -2s \lambda \varphi (\nabla \psi \cdot \nabla) w - 2s \lambda^2 \varphi |\nabla \psi|^2 w, \]
we can write equations (3.5) as
\[ (3.6) \quad \frac{\partial w}{\partial t} + R(x,t)w - P(x,t)w = e^{s\alpha} g - e^{s\alpha} \nabla q \quad \text{in } Q. \]

Multiplying equation (3.6) by itself and integrating over \( Q \), we obtain
\[ (3.7) \quad \int_Q \left( \left| \frac{\partial w}{\partial t} + R(x,t)w - P(x,t)w \right|^2 - |P(x,t)w|^2 \right) dx dt - 2 \int_Q P(x,t)w \cdot R(x,t)w dx dt \]
\[ = 2 \int_Q \frac{\partial w}{\partial t} \cdot P(x,t)w dx dt + \int_Q e^{2s\alpha} |g - \nabla q|^2 dx dt. \]

We set
\[ I = - \int_Q P(x,t)w \cdot R(x,t)w dx dt, \quad J = \int_Q \frac{\partial w}{\partial t} \cdot P(x,t)w dx dt. \]

From (3.7) we have
\[ (3.8) \quad I \leq J + \int_Q e^{2s\alpha} |g|^2 dx dt + \int_Q e^{2s\alpha} |\nabla q|^2 dx dt. \]
This inequality will lead us to the desired final estimate (3.4).

First we deal with $I$. Obviously, $I$ can be written as a sum of ten terms (integrals): $I = \sum_{i=1}^{10} I_i$. Let us examine them by turns.

Two integration by parts and the fact that

$$N = -\frac{\nabla \psi}{|\nabla \psi|}$$

(mind that $\psi = 0$ on $\partial \Omega$) give

$$I_1 \geq -s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla w|^2 dx \, dt - c s \lambda \int_Q \varphi |\nabla w|^2 dx \, dt$$

$$+ 2s\lambda \int_{\Sigma} \varphi |\nabla \psi|^2 \left( \sum_{i=1}^{3} \frac{\partial w_i}{\partial N} \right)^2 \, d\sigma \, dt - s\lambda \int_{\Sigma} \varphi |\nabla \psi|^2 |\nabla w|^2 \, d\sigma \, dt,$$

where $c$ is a positive constant depending only on $\psi$. (For more details we refer to [4].)

Integrating by parts once again (in fact, using Green’s formula), we obtain

$$I_2 = -2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 w \cdot \Delta w \, dx \, dt$$

$$= 2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla w|^2 dx \, dt$$

$$+ 2s\lambda^3 \sum_{i=1}^{3} \int_Q \varphi |\nabla \psi|^2 w_i \nabla \psi \cdot \nabla w_i \, dx \, dt$$

$$+ 4s\lambda^2 \left( \sum_{i=1}^{3} \int_Q \varphi w_i \sum_{j,k=1}^{3} \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_j} \frac{\partial w_i}{\partial x_k} \right) \, dx \, dt$$

$$- 2s\lambda^2 \int_{\Sigma} \varphi |\nabla \psi|^2 \sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_j} N_j \, d\sigma \, dt.$$
\[
4s\lambda^2 \sum_{i=1}^{3} \int_{Q} \varphi w_i \sum_{j,k=1}^{3} \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_j} \frac{\partial w_i}{\partial x_k} \, dx \, dt
\]
\[
\geq -cs\lambda^2 \int_{Q} \varphi |w|^2 \, dx \, dt - \frac{1}{4} s\lambda^2 \int_{Q} \varphi |\nabla \psi|^2 |\nabla w|^2 \, dx \, dt.
\]
Now let us examine the surface integral in (3.11). Using the second boundary condition in (3.5) with \( T = w \) and (3.9), we have
\[
\sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_j} N_j = -s\lambda \varphi |\nabla \psi||w|^2 - \sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_i} N_j
\]
\[
= -s\lambda \varphi |\nabla \psi||w|^2 + |\nabla \psi|^{-1} w \cdot \nabla (w \cdot \nabla \psi)
\]
\[
- |\nabla \psi|^{-1} \sum_{i,j=1}^{3} w_i w_j \frac{\partial^2 \psi}{\partial x_i \partial x_j}.
\]
Since, by (3.9) and the first boundary condition in (3.5), \( w \cdot \nabla \psi = -|\nabla \psi|(w \cdot N) \) = 0 on \( \partial \Omega \), the vectors \( \nabla (w \cdot \nabla \psi) \) and \( N \) have the same direction (at the same point of \( \partial \Omega \)). Combining this remark with the fact that \( w \cdot N = 0 \) on \( \partial \Omega \), we have in (3.14)
\[
w \cdot \nabla (w \cdot \nabla \psi) = 0 \quad \text{on} \quad \partial \Omega.
\]
Thus the surface integral in (3.11) can be written as
\[
-2s\lambda^2 \int_{\Sigma} \varphi |\nabla \psi|^2 \sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_j} N_j \, d\sigma \, dt
\]
\[
= 2s\lambda^3 \int_{\Sigma} \varphi \frac{2}{3} |\nabla \psi|^3 |w|^2 \, d\sigma \, dt - 2s\lambda^2 \int_{\Sigma} \varphi |\nabla \psi| \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} w_i w_j \, d\sigma \, dt.
\]
Let us estimate the second integral in the right–hand side of (3.15) in an adequate manner (see the comments in [4]). We have
\[
-2s\lambda^2 \int_{\Sigma} \varphi |\nabla \psi| \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} w_i w_j \, d\sigma \, dt
\]
\[
\geq -cs\lambda^2 \sum_{i=1}^{3} \int_{0}^{T} \hat{\varphi}(t) \left( \int_{\partial \Omega} w_i^2 \, d\sigma \right) dt.
\]
For $1/2 \leq \alpha < 1$ and $0 < \beta < \alpha$, using (by turns) the trace theorem, an interpolation inequality and Young’s inequality, we obtain

$$s\lambda^2 \int_{\partial\Omega} w_i^2 \, d\sigma \leq c_1 s\lambda^2 |w_i|_{H^{\alpha-rac{1}{2}}(\partial\Omega)}^2 \leq c_2 s\lambda^2 |w_i|_{H^\alpha(\Omega)}^2 \leq c_3 (1 - \alpha)(s\lambda^2)^{\frac{1-\alpha}{2}} |w_i|_{L^2(\Omega)}^2 + c_3 \alpha (s\lambda^2)^{\frac{\beta}{2}} |w_i|_{L^2(\Omega)}^2 + c_3 \alpha (s\lambda^2)^{\frac{\beta}{2}} |\nabla w_i|_{L^2(\Omega)}^2,$$

where $c_1, c_2$ and $c_3$ are positive constants which depend only on $\Omega$. Combining (3.15), (3.16) and (3.17) where $\alpha = 9/16$, $\beta = 1/4$, and inserting the inequality obtained in this way and inequalities (3.12), (3.13) into (3.11), we can derive

$$I_2 \geq \frac{3}{2} s\lambda^2 \int_Q |\nabla \psi|^2 |\nabla w|^2 \, dx \, dt - c \left( s\lambda^4 \int_Q |\nabla |w|^2 |w| \, dx \, dt + s^2 \lambda^2 \int_Q |\nabla |\psi|^2 |\nabla w|^2 \, dx \, dt + s\lambda \int \int_Q |\nabla w|^2 \, dx \, dt \right) + 2s^2 \lambda^3 \int \int_Q |\nabla \psi|^3 |w|^2 \, d\sigma \, dt$$

for $\lambda > 1$,

where, as before, $c$ is a positive constant which depends only on $\psi$.

An integration by parts followed by some calculation leads to the inequality

$$I_3 = -2s^3 \lambda^3 \int_Q \varphi^3 |\nabla \psi|^2 w \cdot (\nabla \psi \cdot \nabla) w \, dx \, dt$$

$$\geq 3s^3 \lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |w|^2 \, dx \, dt - cs^3 \lambda^3 \int_Q \varphi^3 |w|^2 \, dx \, dt - s^3 \lambda^3 \int \int_Q \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial N} |w|^2 \, d\sigma \, dt.$$

We set

$$I_4 = -2s^3 \lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |w|^2 \, dx \, dt.$$

We have as before
\[ I_5 = -2s^2\lambda^3 \int_Q \varphi^2 |\nabla \psi|^2 w \cdot (\nabla \psi \cdot \nabla) w \, dx \, dt \]

\[ \geq 2s^2\lambda^4 \int_Q \varphi^2 |\nabla \psi|^4 |w|^2 \, dx \, dt - cs^2\lambda^3 \int_Q \varphi^2 |w|^2 \, dx \, dt \]

\[ - s^2\lambda^3 \int_\Sigma \varphi^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial N} |w|^2 \, d\sigma \, dt. \]

We also set

\[ I_6 = -2s^2\lambda^4 \int_Q \varphi^2 |\nabla \psi|^4 |w|^2 \, dx \, dt. \]

It is easy to see that

\[ I_7 = 2s^2\lambda^2 \int_Q \varphi^2 \Delta \psi \, w \cdot (\nabla \psi \cdot \nabla) w \, dx \, dt \]

\[ \geq -c\left(s^3\lambda^3 \int_Q \varphi^3 |w|^2 \, dx \, dt + s\lambda \int_Q \varphi |\nabla w|^2 \, dx \, dt\right) \]

and

\[ I_8 = 2s^2\lambda^3 \int_Q \varphi^2 |\nabla \psi|^2 \Delta \psi |w|^2 \, dx \, dt \geq -cs^2\lambda^3 \int_Q \varphi^2 |w|^2 \, dx \, dt. \]

All constants \(c\) in (3.19), (3.21), (3.23), (3.24) are positive and depend only on \(\psi\).

Finally, integrating by parts, we obtain after some calculation

\[ I_9 = 2s^2\lambda \int_Q \varphi \frac{\partial \alpha}{\partial t} \, w \cdot (\nabla \psi \cdot \nabla) w \, dx \, dt \]

\[ - 2s^2\lambda^2 \int_Q \varphi |\nabla \psi|^2 \frac{\partial \alpha}{\partial t} |w|^2 \, dx \, dt \]

\[ + 8s^2\lambda^2 \gamma(\lambda) \int_Q \varphi |\nabla \psi|^2 \frac{1}{(T-t)^9} (T-2t)|w|^2 \, dx \, dt \]

\[ - s^2\lambda \int_Q \varphi \frac{\partial \alpha}{\partial t} \Delta \psi |w|^2 \, dx \, dt \]

\[ + s^2\lambda \int_\Sigma \varphi \frac{\partial \alpha}{\partial t} \frac{\partial \psi}{\partial N} |w|^2 \, d\sigma \, dt, \]

where

\[ \gamma(\lambda) = e^{2\lambda |\psi|_{L^2(\Omega)}}. \]
Adding $I_9$ to the integral

$$I_{10} = 2s^2 \lambda^2 \int_Q \varphi |\nabla \psi|^2 \frac{\partial \alpha}{\partial t} |w|^2 dx \, dt,$$

we have

$$I_9 + I_{10} \geq -cs^2 \lambda^2 \gamma(\lambda) \int_Q \varphi \frac{\partial^2 \psi}{\partial t^2} |w|^2 dx \, dt$$

(3.25)

$$+ s^2 \lambda \int_Q \frac{\partial \alpha}{\partial t} \frac{\partial \psi}{\partial N} |w|^2 d\sigma \, dt$$

for $\lambda > 1,$

where $c$ is a positive constant depending on $\psi$ and $T.$

Now we shall estimate the integral $J.$ By Green's formula, an integration by parts with respect to $t$ and the boundary conditions in (3.5) (in fact (3.14) where the factor $w_i$ is replaced by $\partial w_i/\partial t),$ we have

$$- \int_Q \Delta w \cdot \frac{\partial w}{\partial t} dx \, dt$$

$$= \frac{1}{2} \int_Q \frac{\partial}{\partial t} |\nabla w|^2 dx \, dt - \sum_{i=1}^3 \int_{\Sigma} \frac{\partial w_i}{\partial N} \frac{\partial w}{\partial t} d\sigma \, dt$$

$$= - \int_{\Sigma} \sum_{i,j=1}^3 \frac{\partial w_i}{\partial x_j} \frac{\partial w}{\partial t} N_j d\sigma \, dt$$

(3.26)

$$= \frac{1}{2} s \lambda \int_Q \varphi |\nabla \psi| \frac{\partial}{\partial t} |w|^2 d\sigma \, dt + \int_{\Sigma} \sum_{i,j=1}^3 \frac{\partial w_j}{\partial x_i} \frac{\partial w}{\partial t} N_j d\sigma \, dt$$

$$= - \frac{1}{2} s \lambda \int_{\Sigma} \frac{\partial \varphi}{\partial t} \frac{1}{|\nabla \psi|} |w|^2 d\sigma \, dt - \int_{\Sigma} \frac{1}{|\nabla \psi|} \nabla (w \cdot \nabla \psi) \cdot \frac{\partial w}{\partial t} d\sigma \, dt$$

$$+ \frac{1}{2} \int_{\Sigma} \frac{1}{|\nabla \psi|} \sum_{i,j=1}^3 \frac{\partial^2 \psi}{\partial x_i \partial x_j} \frac{\partial}{\partial t} (w_i w_j) d\sigma \, dt$$

$$= - \frac{1}{2} s \lambda \int_{\Sigma} \frac{\partial \varphi}{\partial t} |\nabla \psi| |w|^2 d\sigma \, dt.$$

Some integrations by parts with respect to $t$ together with (3.25) give
\[ J = \frac{1}{2} s \lambda \left[ \int_{\Sigma} \frac{\partial \varphi}{\partial t} |\nabla \psi||w|^2 d\sigma \right] dt \]
\[ + s^2 \lambda^2 \left[ \int_Q \varphi \frac{\partial \varphi}{\partial t} |\nabla \psi|^2|w|^2 dx \right] dt + \frac{1}{2} s \lambda^2 \left[ \int_Q \frac{\partial \varphi}{\partial t} |\nabla \psi|^2|w|^2 dx \right] dt \]
\[ - \frac{1}{2} s \lambda \left[ \int_Q \frac{\partial \varphi}{\partial t} \Delta \psi |w|^2 dx \right] dt - \frac{1}{2} s \int_Q \frac{\partial^2 \alpha}{\partial t^2} |w|^2 dx dt \]
\[ \leq - \frac{1}{2} s \lambda \left[ \int_{\Sigma} \frac{\partial \varphi}{\partial t} |\nabla \psi||w|^2 d\sigma \right] dt \]
\[ + c \left( s^2 \lambda^2 \int_Q \varphi \frac{17}{8} |w|^2 dx dt + s(1 + \gamma(\lambda)) \int_Q \varphi \frac{5}{4} |w|^2 dx dt \right) \]
\[ \text{for } s > 1, \lambda > 1, \]

where \( c \) is a positive constant depending on \( \psi \) and \( T \).

Before inserting the above estimates for \( I_1, \ldots, I_{10} \) and \( J \) in inequality (3.8), we must eliminate the surface integrals in (3.10), (3.18), (3.19), (3.21), (3.25) and (3.27). (Note that all these integrals are multiplied by powers of \( \lambda \) with odd exponents.) To get rid of the surface integrals, we have to repeat all the considerations and calculations before but for \( \varpi = e^{s\alpha} z \) instead of \( w \).

Thus, changing \( z \) by \( \varpi \) in (3.1), we obtain
\[
\frac{\partial \varpi}{\partial t} + R(x, t)\varpi - P(x, t)\varpi = e^{s\alpha} g - e^{s\alpha} \nabla q \quad \text{in } Q,
\]

where
\[
P(x, t)\varpi = -\Delta \varpi - s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 \varpi - s \lambda^2 \varphi |\nabla \psi|^2 \varpi - s \lambda \varphi \Delta \psi \varpi + s \frac{\partial \alpha}{\partial t} \varpi,
\]
\[
Q(x, t)\varpi = 2s \lambda \varphi (\nabla \psi \cdot \nabla) \varpi - 2s \lambda^2 \varphi |\nabla \psi|^2 \varpi.
\]

In the same way as (3.8) is obtained, we have
\[ I \leq J + \int_Q e^{2s\alpha} |g|^2 dx dt + \int_Q e^{2s\alpha} |\nabla q|^2 dx dt, \]

where
\[ I = - \int_Q P(x, t)\varpi \cdot R(x, t)\varpi dx dt, \quad J = \int_Q \frac{\partial \varpi}{\partial t} \cdot P(x, t)\varpi dx dt. \]
We denote by $I_1, \ldots, I_{10}$ the ten integrals contained in $I = \sum_{i=1}^{10} I_i$. Doing similar calculations as before, we obtain

\[
T_1 \geq -s\lambda^2 \int_Q \varphi|\nabla \psi|^2|\nabla w|^2 dx \, dt - cs\lambda \int_Q \varphi|\nabla w|^2 dx \, dt
- 2s\lambda \int_\Sigma \varphi|\nabla \psi||\nabla w|^2 d\sigma \, dt, \tag{3.29}
\]

\[
T_2 \geq \frac{3}{2} s\lambda^2 \int_Q \varphi|\nabla \psi|^2|\nabla w|^2 dx \, dt
- c\left(s\lambda^4 \int_Q \varphi|\nabla w|^2 dx \, dt + s^2 \lambda^2 \int_Q \varphi|\nabla w|^2 dx \, dt + s\lambda \int_Q \varphi|\nabla w|^2 dx \, dt\right)
- 2s^2 \lambda^3 \int_\Sigma \varphi^2|\nabla \psi|^3|w|^2 d\sigma \, dt \quad \text{for } \lambda > 1, \tag{3.30}
\]

\[
T_3 + T_4 \geq s^4 \lambda^4 \int_Q \varphi^3|\nabla \psi|^4|w|^2 dx \, dt
- 2s^3 \lambda^3 \int_Q \varphi^3|\nabla w|^2 dx \, dt + s^3 \lambda^3 \int_\Sigma \varphi^3|\nabla \psi|\frac{\partial \psi}{\partial N}|w|^2 d\sigma \, dt, \tag{3.31}
\]

\[
T_5 + T_6 \geq -cs^2 \lambda^3 \int_Q \varphi^3|\nabla w|^2 dx \, dt + s^2 \lambda^3 \int_\Sigma \varphi^2|\nabla \psi|^2 \frac{\partial \psi}{\partial N}|w|^2 d\sigma \, dt, \tag{3.32}
\]

\[
T_7 + T_8 \geq -c\left(s^3 \lambda^3 \int_Q \varphi^3|\nabla w|^2 dx \, dt + s^2 \lambda^3 \int_Q \varphi^2|\nabla w|^2 dx \, dt\right)
+ s\lambda \int_Q \varphi|\nabla w|^2 dx \, dt, \tag{3.33}
\]

\[
T_9 + T_{10} \geq -cs^2 \lambda^2 \gamma(\lambda) \int_Q \varphi^2|\nabla w|^2 dx \, dt - s^2 \lambda \int_\Sigma \varphi\frac{\partial \alpha}{\partial t}\frac{\partial \psi}{\partial N}|w|^2 d\sigma \, dt \quad \text{for } \lambda > 1, \tag{3.34}
\]

and

\[
J \leq \frac{1}{2} s\lambda \int_\Sigma \frac{\partial \varphi}{\partial t}|\nabla \psi||w|^2 d\sigma \, dt
+ c\left(s^2 \lambda^2 \int_Q \varphi^2|\nabla w|^2 dx \, dt + \gamma(\lambda) s(1 + \gamma(\lambda)) \int_Q \varphi^2|\nabla w|^2 dx \, dt\right) \quad \text{for } s > 1, \lambda > 1, \tag{3.35}
\]
where $c$ is a positive constant depending on $\psi$ and $T$. Here we have used the fact that
\[ \varphi = \varphi, \quad \alpha = \alpha, \quad \text{and} \quad \varpi = w \text{ on } \Sigma \]
(because $\psi = 0$ on $\Sigma$). We also note that
\[ (3.36) \quad \varphi \leq \varphi, \quad \alpha \leq \alpha, \quad \text{and} \quad |\varpi| \leq |w| \text{ in } Q. \]

Moreover, we have
\[ \frac{\partial w}{\partial x_i} = e^{s(\pi - \alpha)} \frac{\partial w}{\partial x_i} + s \left( \frac{\partial \alpha}{\partial x_i} - \frac{\partial \alpha}{\partial x_i} \right) \varpi, \]
whence it immediately follows that
\[ (3.37) \quad \frac{\partial \varpi}{\partial x_i} = \frac{\partial w}{\partial x_i} - 2s\lambda \frac{\partial \psi}{\partial x_i} w \text{ on } \Sigma \]
and
\[ (3.38) \quad |\nabla \varpi| \leq \sqrt{2}|\nabla w| + 2\sqrt{2} s\lambda \varphi |\nabla \psi||w| \text{ in } Q. \]

Let us estimate the sum of the four surface integrals in (3.10) and (3.29) by integrals over $Q$ of $|w|^2$ and $|\nabla w|^2$ multiplied by convenient powers of $s, \lambda$ and $\varphi$. We set
\[ S = 2s\lambda \int_{\Sigma} \varphi |\nabla \psi|^2 \sum_{i=1}^3 \left( \frac{\partial w_i}{\partial N} \right)^2 d\sigma dt - s\lambda \int_{\Sigma} \varphi |\nabla \psi||\nabla w|^2 d\sigma dt, \]
\[ \mathcal{S} = -2s\lambda \int_{\Sigma} \varphi |\nabla \psi|^2 \sum_{i=1}^3 \left( \frac{\partial \varpi_i}{\partial N} \right)^2 d\sigma dt + s\lambda \int_{\Sigma} \varphi |\nabla \psi||\nabla \varpi|^2 d\sigma dt. \]

By (3.37) we can write
\[ \sum_{i=1}^3 \left( \frac{\partial \varpi_i}{\partial N} \right)^2 = \sum_{i=1}^3 \left( \frac{\partial w_i}{\partial N} \right)^2 + 4s\lambda \varphi |\nabla \psi|^2 \sum_{i=1}^3 w_i \frac{\partial w_i}{\partial N} + 4s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 |w|^2 \text{ on } \Sigma, \]
\[ |\nabla \varpi|^2 = |\nabla w|^2 + 4s\lambda \varphi |\nabla \psi|^2 \sum_{i=1}^3 w_i \frac{\partial w_i}{\partial x_i} N_j + 4s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 |w|^2 \text{ on } \Sigma. \]
Inserting these expressions into that of \( S \), we have
\[
S + \mathcal{S} = -4s^2 \lambda^2 \int_{\Sigma} \varphi^2 |\nabla \psi|^2 \sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_j} N_j \, d\sigma \, dt - 4s^3 \lambda^3 \int_{\Sigma} \varphi^3 |\nabla \psi|^3 |w|^2 \, d\sigma \, dt.
\]

Rewriting \( \sum_{i,j=1}^{3} w_i \frac{\partial w_i}{\partial x_j} N_j \) by using (3.14), we obtain
\[
S + \mathcal{S} = -4s^2 \lambda^2 \int_{\Sigma} \varphi^2 |\nabla \psi|^2 \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} w_i w_j \, d\sigma \, dt.
\]

This last expression of \( S + \mathcal{S} \) can be treated in the same manner as the similar one in (3.15) by using inequality (3.17), in this case with \( \alpha = 1/2 \), \( \beta = 1/4 \), and \( s \) replaced by \( s^2 \hat{\varphi}^2(t) \). Thus we obtain
\[
(3.39) \quad S + \mathcal{S} \geq -c \left( s^3 \lambda^3 \int_{Q} \varphi^3 |w|^2 \, dx \, dt + s\lambda \int_{Q} \varphi |\nabla w|^2 \, dx \, dt \right),
\]
where \( c \) is a positive constant depending only on \( \psi \).

Now we add inequalities (3.8) and (3.28), and then use estimates (3.10), (3.18) through (3.25), (3.27), (3.29) through (3.35) and (3.39). (We remind that \( \mathcal{I} = \sum_{i=1}^{10} I_i \) and \( \mathcal{I} = \sum_{i=1}^{10} I_i \).) Replacing \( \varphi, |w|, |\nabla w|, \) and \( \sigma \) in the right–hand side by \( \varphi, |w|, |\nabla w|, \) and \( \alpha \) by means of inequalities (3.36) and (3.38), we obtain
\[
(3.40) \quad s\lambda^2 \int_{Q} \varphi |\nabla \psi|^2 |\nabla w|^2 \, dx \, dt + s^3 \lambda^4 \int_{Q} \varphi^3 |\nabla \psi|^4 |w|^2 \, dx \, dt
\leq c \left( s\lambda \int_{Q} \varphi |\nabla w|^2 \, dx \, dt + s^3 \lambda^3 \int_{Q} \varphi^3 |w|^2 \, dx \, dt + s^2 \lambda^4 \gamma(\lambda) \int_{Q} \varphi^3 |w|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} |g|^2 \, dx \, dt \right) \quad \text{for } s > 1, \lambda > 1,
\]
where the constant \( c \) depends only on \( \psi \) and \( T \).

The crucial point of the proof is now coming. We shall form integrals over \( Q_{\omega_0} \) of \( \varphi |\nabla w|^2 \) and \( \varphi^3 |w|^2 \) in the right–hand side of our inequality by
suitable subtractions, which are possible thanks to the fact that, on the one hand, \(|\nabla \psi| > 0\) in \(\overline{\Omega} \setminus \omega_0\) and, on the other hand, the powers of \(\lambda\) and \(s\) corresponding to similar terms in the two sides of inequality (3.40) are greater in the left–hand side than in the other one. So, by (3.2), \(|\nabla \psi| \geq \rho_0\) in \(\overline{\Omega} \setminus \omega_0\) for some \(\rho_0 > 0\). This allows us to derive from inequality (3.40) the following one:

\[
(3.41) \quad s\lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla w|^2 dx \, dt + s^3 \lambda^4 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |w|^2 dx \, dt 
\leq c \left( s\lambda \int_Q \varphi |\nabla w|^2 dx \, dt + s^3 \lambda^3 \int_Q \varphi^3 |w|^2 dx \, dt 
+ s^2 \lambda^4 \gamma(\lambda) \int_Q \varphi^3 |w|^2 dx \, dt + \int_Q e^{2s\alpha} |g|^2 dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 dx \, dt \right),
\]

where \(c\) is the constant in (3.40) multiplied by \((\min(\rho_0^2, \rho_0^4))^{-1}\). Now, we take

\[
\lambda > \lambda_0 = c + 1
\]

to have \(s\lambda^2 - cs\lambda > s\lambda\) and \(\lambda - c - 1 > 0\), where \(c\) is the constant in (3.41). Then taking

\[
s > s_0(\lambda) = \max \left( \frac{c\lambda \gamma(\lambda)}{\lambda - c - 1}, 1 \right),
\]

we have \(s^3 \lambda^4 - cs^3 \lambda^3 - cs^2 \lambda^4 \gamma(\lambda) > s^3 \lambda^3\), and so

\[
s\lambda \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla w|^2 dx \, dt + s^3 \lambda^3 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |w|^2 dx \, dt 
\leq s\lambda \int_{Q_{\omega_0}} \varphi |\nabla w|^2 dx \, dt + (1 + \lambda \gamma(\lambda)) s^3 \lambda^3 \int_{Q_{\omega_0}} \varphi^3 |w|^2 dx \, dt 
+ \int_Q e^{2s\alpha} |g|^2 dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 dx \, dt.
\]

Finally, adding

\[
s\lambda \int_{Q_{\omega_0}} \varphi |\nabla w|^2 dx \, dt + s^3 \lambda^3 \int_{Q_{\omega_0}} \varphi^3 |w|^2 dx \, dt
\]

to both sides of the preceding inequality, we obtain
\[
\begin{align*}
\int_{Q_{\omega_0}} e^{2s\alpha} |\nabla z|^2 dx dt &+ \int_{Q_{\omega_1}} e^{2s\alpha} |g|^2 dx dt + \int_{Q} e^{2s\alpha} |\nabla q|^2 dx dt \\
\leq c(\lambda) \left( \int_{Q_{\omega_0}} e^{2s\alpha} |\nabla z|^2 dx dt + \int_{Q_{\omega_1}} e^{2s\alpha} |z|^2 dx dt + \int_{Q} e^{2s\alpha} |g|^2 dx dt + \int_{Q} e^{2s\alpha} |\nabla q|^2 dx dt \right)
\end{align*}
\]
for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \),

where \( c(\lambda) = 2\lambda^3(1 + \lambda \gamma(\lambda)) \).

Coming back to \( z (w = e^{s\alpha}z) \), we can rewrite inequality (3.42) as

\[
\begin{align*}
\int_{Q_{\omega_0}} e^{2s\alpha} |\nabla z|^2 dx dt &+ \int_{Q_{\omega_1}} e^{2s\alpha} |z|^2 dx dt \\
\leq c(\lambda) \left( \int_{Q_{\omega_0}} e^{2s\alpha} |\nabla z|^2 dx dt + \int_{Q_{\omega_1}} e^{2s\alpha} |z|^2 dx dt + \int_{Q} e^{2s\alpha} |g|^2 dx dt \right)
\end{align*}
\]
for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \),

where \( \lambda_0 \) is possibly greater than \( \lambda_0 \) before and \( c(\lambda) \) is also somehow greater than that before.

The integral of \( e^{2s\alpha} |\nabla z|^2 \) over \( Q_{\omega_0} \) in the right-hand side of (3.43) can be estimate (in its turn) by the integral of \( e^{2s\alpha} |z|^2 \) over a somehow larger domain \( Q_{\omega_1} = \omega_1 \times (0, T) \) and the last two integrals in the right-hand side of (3.43). To this end, we have to multiply equation (3.1) by \( \chi \varphi e^{2s\alpha} z \) and integrate over \( Q_0 \), where \( \chi \) is some function in \( C_0^{\infty}(\Omega) \) which is 1 in \( \omega_0 \) and 0 in \( \Omega \setminus \omega_1 \), for some open subset \( \omega_1 \) of \( \omega \) such that \( \omega_0 \subset \subset \omega_1 \subset \subset \omega \) (\( \omega_1 \) can be that in the statement of Theorem 3.1). Doing the calculations exactly as in the proof of Theorem 3.2 in [4] (see also [3]), we finally obtain

\[
\int_{Q_{\omega_0}} e^{2s\alpha} |\nabla z|^2 dx dt \\
\leq c(\lambda) \left( s \int_{Q_{\omega_1}} e^{2s\alpha} |z|^2 dx dt + \frac{1}{s} \int_{Q} e^{2s\alpha} |g|^2 dx dt + \frac{1}{s} \int_{Q} e^{2s\alpha} |\nabla q|^2 dx dt \right)
\]
for \( \lambda \) large enough and \( s > 1 \).
Coupling inequalities (3.43) and (3.44), we have

\[
(3.45) \quad s\lambda \int_Q e^{2s\alpha} \varphi |\nabla z|^2 \, dx \, dt + s^3 \lambda^3 \int_Q e^{2s\alpha} \varphi^3 |z|^2 \, dx \, dt \\
\leq c(\lambda) \left( \int_Q e^{2s\alpha} s^3 \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha} |g|^2 \, dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 \, dx \, dt \right)
\]

for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \),

with \( \lambda_0 \) possibly greater than that in (3.43).

Now, let us obtain estimates similar to (3.45) also for the first and second order derivatives of \( z \) with respect to the time and space variables, respectively. First we shall estimate the integrals of \( e^{2s\alpha} \varphi^{-1} |\partial z/\partial t|^2 \) and \( e^{2s\alpha} \varphi^{-1} |\Delta z|^2 \) over \( Q \) by the integral of \( e^{2s\alpha} |\nabla z|^2 \). To this aim we multiply equation (3.1) by \( e^{2s\alpha} \varphi^{-1} \partial z/\partial t \) and integrate over \( Q \). So, we have

\[
(3.46) \quad \int_Q e^{2s\alpha} \varphi^{-1} \left| \frac{\partial z}{\partial t} \right|^2 \, dx \, dt \\
= - \int_Q e^{2s\alpha} \varphi^{-1} \frac{\partial z}{\partial t} \cdot \Delta z \, dx \, dt + \int_Q e^{2s\alpha} \varphi^{-1} \frac{\partial z}{\partial t} \cdot (g - \nabla q) \, dx \, dt.
\]

Using a version of Green’s formula for vector functions (see [11], p. 53) together with the boundary conditions in (3.1) (mind that \( N \cdot \partial z/\partial t = 0 \) on \( \Sigma \)) and then integrating by parts with respect to \( t \), we successively obtain

\[
\int_Q e^{2s\alpha} \varphi^{-1} \frac{\partial z}{\partial t} \cdot \Delta z \, dx \, dt \\
= - \frac{1}{2} \sum_{i,j=1}^3 \int_Q \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) \left( \frac{\partial}{\partial x_j} \left( e^{2s\alpha} \varphi^{-1} \frac{\partial z_i}{\partial t} \right) \right) dx \, dt \\
+ \frac{\partial}{\partial x_i} \left( e^{2s\alpha} \varphi^{-1} \frac{\partial z_j}{\partial t} \right) \right) dx \, dt \\
= - \frac{1}{2} \sum_{i,j=1}^3 \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) \left( \frac{\partial \psi}{\partial x_j} \frac{\partial z_i}{\partial t} + \frac{\partial \psi}{\partial x_i} \frac{\partial z_j}{\partial t} \right) dx \, dt \\
+ \frac{\lambda}{2} \sum_{i,j=1}^3 \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) \left( \frac{\partial \psi}{\partial x_j} \frac{\partial z_i}{\partial t} + \frac{\partial \psi}{\partial x_i} \frac{\partial z_j}{\partial t} \right) dx \, dt.
\]
Then, it is easy to see that:

\[ \frac{1}{4} \int_Q e^{2s\alpha} \varphi^{-2} \frac{\partial \varphi}{\partial t} \sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right)^2 dx dt \leq \frac{T^{31}}{2^{27}} \int_Q e^{2s\alpha} |\nabla z|^2 dx dt, \]

(3.48) \[ - \frac{1}{2} s \int_Q e^{2s\alpha} \varphi^{-1} \frac{\partial \alpha}{\partial t} \sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right)^2 dx dt \]

\[ \leq \frac{T^{15}}{2^{10}} (1 + \gamma(\lambda)) s \int_Q e^{2s\alpha} |\nabla z|^2 dx dt, \]

(3.49) \[ - \frac{1}{2} \int_Q e^{2s\alpha} \varphi^{-1} \sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) \left( \frac{\partial \psi}{\partial x_j} \frac{\partial z_i}{\partial t} + \frac{\partial \psi}{\partial x_i} \frac{\partial z_j}{\partial t} \right) dx dt \]

\[ \leq 18 \left( \frac{T}{2} \right)^{32} \lambda^2 \sup_{\Omega} |\nabla \psi|^2 \int_Q e^{2s\alpha} |\nabla z|^2 dx dt + \frac{1}{6} \int_Q e^{2s\alpha} \varphi^{-1} |\frac{\partial z}{\partial t}|^2 dx dt, \]

(3.50) \[ \lambda s \int_Q e^{2s\alpha} \sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) \left( \frac{\partial \psi}{\partial x_j} \frac{\partial z_i}{\partial t} + \frac{\partial \psi}{\partial x_i} \frac{\partial z_j}{\partial t} \right) dx dt \]

\[ \leq 72 \lambda^2 \sup_{\Omega} |\nabla \psi|^2 \int_Q e^{2s\alpha} |\nabla z|^2 dx dt + \frac{1}{6} \int_Q e^{2s\alpha} \varphi^{-1} |\frac{\partial z}{\partial t}|^2 dx dt, \]

(3.51) \[ \int_Q e^{2s\alpha} \varphi^{-1} \frac{\partial z}{\partial t} \cdot (g - \nabla q) dx dt \]

\[ \leq 3 \left( \frac{T}{2} \right)^{16} \int_Q e^{2s\alpha} |g|^2 dx dt + 3 \left( \frac{T}{2} \right)^{16} \int_Q e^{2s\alpha} |\nabla q|^2 dx dt \]

\[ + \frac{1}{6} \int_Q e^{2s\alpha} \varphi^{-1} |\frac{\partial z}{\partial t}|^2 dx dt. \]

(3.52)

(Here we have used the obvious inequalities: \[ \sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right)^2 \leq 4 |\nabla z|^2, \]

\[ 1 \leq (T/2)^{16} \varphi, \varphi^{1/8} \leq (T/2)^{14} \varphi, \text{ and } \varphi^{-1} \leq (T/2)^{32} \varphi. \] Thus (3.46), in conjunction with (3.47) and inequalities (3.48) through (3.52), yields

\[ \int_Q e^{2s\alpha} (s\varphi)^{-1} |\frac{\partial z}{\partial t}|^2 dx dt \]

\[ \leq c(\lambda) \left( s \int_Q e^{2s\alpha} |\nabla z|^2 dx dt + \int_Q e^{2s\alpha} |g|^2 dx dt + \int_Q e^{2s\alpha} |\nabla q|^2 dx dt \right) \text{ for } s > 1, \]
which together with (3.45) gives

\[
\int_Q e^{2s\alpha}(s\varphi)^{-1} \left| \frac{\partial z}{\partial t} \right|^2 \, dx \, dt
\]

(3.53)

\[
\leq c(\lambda) \left( 3 \int_{Q_{\omega_1}} e^{2s\alpha} \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha} |g|^2 \, dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 \, dx \, dt \right)
\]

for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \).

Next multiplying equation (3.1) by \( e^{2s\alpha}\varphi^{-1}\Delta z \) and integrating over \( Q \), we obtain

\[
\int_Q e^{2s\alpha}\varphi^{-1} |\Delta z|^2 \, dx \, dt
\]

\[
= - \int_Q e^{2s\alpha}\varphi^{-1} \Delta z \cdot \frac{\partial z}{\partial t} \, dx \, dt + \int_Q e^{2s\alpha}\varphi^{-1} \Delta z \cdot (g - \nabla q) \, dx \, dt,
\]

which, after some arrangements and the use of (3.53), yields

\[
\int_Q e^{2s\alpha}(s\varphi)^{-1} |\Delta z|^2 \, dx \, dt
\]

(3.54)

\[
\leq c(\lambda) \left( 3 \int_{Q_{\omega_1}} e^{2s\alpha} \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha} |g|^2 \, dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 \, dx \, dt \right)
\]

for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \).

Now it has only remained to obtain an estimate for \( \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \) similar to that for \( |\Delta z|^2 \) before. But it is fairly transparent that such an estimate could be obtained by using the \( H^2 \), a priori estimate for the solutions of the stationary Stokes equations with the boundary conditions in (3.1), that is, the following one:

\[
|z(\cdot, t)|_{H^2(\Omega)^3} \leq c|\Delta z(\cdot, t)|_{L^2(\Omega)^3}.
\]

However, to embody this still vague idea, we first need to pass to the new \( \tilde{z} = e^{s\alpha}(s\varphi)^{-1/2}z \), because of the weight function \( e^{2s\alpha}(s\varphi)^{-1} \) before

\[
\sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \]

in the estimate we wish to obtain. After some careful calculations led by the idea suggested before (we refer the reader to the proof
of Theorem 3.2 in [4] for more details), we have
\[
\int_Q e^{2s\alpha} (s\varphi)^{-1} \sum_{i,j=1}^{3} \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \, dx \, dt
\leq c(\lambda) \left( \int_Q e^{2s\alpha} s^3 \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha} |\nabla z|^2 \, dx \, dt + \int_Q e^{2s\alpha} (s\varphi)^{-1} |\Delta z|^2 \, dx \, dt \right),
\]
which combined with inequalities (3.45) and (3.54) gives
\[
\int_Q e^{2s\alpha} (s\varphi)^{-1} \sum_{i,j=1}^{3} \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \, dx \, dt
\leq c(\lambda) \left( \int_{Q_{\omega_1}} e^{2s\alpha} s^3 \varphi^3 |z|^2 \, dx \, dt + \int_Q e^{2s\alpha} |g|^2 \, dx \, dt + \int_Q e^{2s\alpha} |\nabla q|^2 \, dx \, dt \right)
\]
for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda) \).

Thus, inequality (3.4) follows by taking (3.45), (3.53), and (3.55) altogether, and so Lemma 3.1 is proved.

2. The estimate for \( q \). The following lemma shows us how the integral over \( Q \) of \( e^{2s\alpha} |\nabla q|^2 \) in (3.4) can be replaced by a similar integral of \( e^{2s\alpha} |q|^2 \) taken over \( Q_{\omega_1} \).

**Lemma 3.2.** Let \( \Omega, \omega, \omega_0, \) and \( \omega_1 \) be open subsets of \( \mathbb{R}^3 \) as in the statement of Theorem 3.1. Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda > \lambda_0 \) one can find \( s_0(\lambda) > 0 \) and \( c(\lambda) > 0 \) such that for \( s > s_0(\lambda) \) we have
\[
\int_Q e^{2s\alpha} |\nabla q|^2 \, dx \, dt
\leq c(\lambda) \left( \int_Q e^{2s\alpha} s^{\frac{11}{4}} \varphi^3 |z|^2 \, dx \, dt + \int_{Q_{\omega_1}} e^{2s\alpha} s^{\frac{11}{4}} \varphi^3 q^2 \, dx \, dt \right.
+ \int_Q e^{2s\alpha} s^{\frac{1}{2}} \varphi^3 |g|^2 \, dx \, dt + \int_Q e^{2s\alpha} s^{\frac{1}{2}} \varphi^3 (\text{div } g)^2 \, dx \, dt \right)
\]
for all \( g \in L^2(0, T; (H^1(\Omega))^3) \) and all corresponding solutions \((z, q) \in (H^{2,1}(Q))^3 \times L(0, T; H^1(\Omega)) \) of system (3.1).

**Proof.** Applying the divergence operator to both sides of equation (3.1), one has
\[
\Delta q = \text{div } g \quad \text{in } Q.
\]
In the paper [10], Imanuvilov obtains a remarkable Carleman–type estimate for the solutions of the (linear) second order uniformly elliptic equations (in divergence form) with nonhomogeneous Dirichlet boundary conditions in domains whose boundaries are only finite unions of mutually disjoint 2–dimensional compact connected manifolds of class $C^2$ (like in the statement of Theorem 3.1). For equation (3.57) this estimate looks as follows: one has some $\lambda_0 > 0$ and, for $\lambda > \lambda_0$, some $s_0(\lambda) > 0$ and $c(\lambda) > 0$ such that

$$\int_\Omega e^{2s\phi} \left( s^\frac{3}{2} \delta^\frac{3}{2} q^2 + s^{-\frac{3}{4}} \delta^{-\frac{3}{4}} |\nabla q|^2 \right) \, dx$$

\begin{equation}
(3.58)
\leq c(\lambda) \left( e^{2s\phi} |q|_{H^\frac{1}{2} (\partial \Omega)}^2 + \int_{\omega_1} e^{2s\phi} s^2 \delta^2 q^2 \, dx + \int_\Omega e^{2s\phi} s^{-\frac{3}{4}} \delta^{-\frac{3}{4}} (\text{div } g) \, dx \right)
\quad \text{a.e. } t \in (0, T), \quad \text{for } \lambda > \lambda_0 \text{ and } s > s_0(\lambda).
\end{equation}

After multiplication by $s^\frac{3}{2} \delta^\frac{3}{2} e^{-2s\gamma(\lambda)/(t(T-t))}$ and integration over $(0, T)$, inequality (3.58) becomes

\begin{equation}
(3.59)
\int_Q e^{2s\phi} |\nabla q|^2 \, dx \, dt
\leq c(\lambda) \left( \int_0^T e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} |q|_{H^\frac{1}{2} (\partial \Omega)} \, dt + \int_{Q_{-1}} e^{2s\phi} s^\frac{11}{4} \delta^\frac{11}{4} q^2 \, dx \, dt + \int_Q e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} (\text{div } g)^2 \, dx \, dt \right)
\quad \text{for } \lambda > \lambda_0 \text{ and } s > s_0(\lambda).
\end{equation}

So, it remains to estimate the first integral in the right–hand side of (3.59) (which involves only the values of $q$ taken on $\Sigma$).

By the trace theorem, we can write

\begin{equation}
(3.60)
\int_0^T e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} |q|_{H^\frac{1}{2} (\partial \Omega)} \, dt \leq c \int_Q e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} (|q|^2 + |\nabla q|^2) \, dx \, dt.
\end{equation}

Multiplying inequality (3.58) by $s^{-\frac{3}{4}} \delta^{-\frac{3}{4}} e^{-2s\gamma(\lambda)/(t(T-t))}$ and integrating over $(0, T)$, we have

\begin{equation}
(3.61)
\int_Q e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} |q|^2 \, dx \, dt
\leq c(\lambda) \left( \int_0^T e^{2s\phi} s^{-\frac{3}{4}} \delta^{-\frac{3}{4}} |q|_{H^\frac{1}{2} (\partial \Omega)}^2 \, dt + \int_{Q_{-1}} e^{2s\phi} s^\frac{3}{4} \delta^\frac{3}{4} q^2 \, dx \, dt + \int_Q e^{2s\phi} s^{-\frac{3}{4}} \delta^{-\frac{3}{4}} (\text{div } g)^2 \, dx \, dt \right)
\quad \text{for } \lambda > \lambda_0 \text{ and } s > s_0(\lambda).
\end{equation}
Now, to estimate $\int_Q e^{2s\hat{\alpha}} s^\frac{3}{4} \hat{\phi}^\frac{3}{4} |\nabla q|^2 \, dx \, dt$, we pass from $z$ and $q$ to $\tilde{z} = e^{s\hat{\alpha}} \hat{\phi}^\frac{3}{8} z$ and $\tilde{q} = e^{s\hat{\alpha}} \hat{\phi}^\frac{3}{8} q$. Since $z$ and $q$ satisfy (3.1), it is easy to see that $\tilde{z}$ and $\tilde{q}$ satisfy

\[
\frac{\partial \tilde{z}}{\partial t} + \Delta \tilde{z} + \nabla \tilde{q} = e^{s\hat{\alpha}} \hat{\phi}^\frac{3}{8} g + e^{s\hat{\alpha}} \left( \frac{3}{8} \hat{\phi}^{-\frac{3}{8}} \hat{\phi}' + s\hat{\phi}^\frac{3}{8} \hat{\alpha}' \right) z \quad \text{in} \ Q,
\]

\[
\text{div} \tilde{z} = 0 \quad \text{in} \ Q,
\]

\[
\tilde{z} \cdot N = 0 \quad \text{on} \ \Sigma,
\]

\[
\sum_{i,j=1}^3 \left( \frac{\partial \tilde{z}_i}{\partial x_j} + \frac{\partial \tilde{z}_j}{\partial x_i} \right) N_i T_j = 0 \quad \text{on} \ \Sigma,
\]

for all tangential vector fields $T = (T_1, T_2, T_3)$ on $\partial \Omega$.

Using the energy a priori estimate for the solutions of the Stokes equations with the boundary conditions in (3.1) (see [13], [14], [15] and [6] for related cases), we have

\[
\int_Q |\nabla \tilde{q}|^2 \, dx \, dt \leq c(\lambda) \left( \int_Q e^{2s\hat{\alpha}} \hat{\phi}^\frac{3}{4} |g|^2 \, dx \, dt + \int_Q e^{2s\hat{\alpha}} s^\frac{1}{2} \hat{\phi}^\alpha |z|^2 \, dx \, dt \right)
\]

(because \( \frac{3}{8} \hat{\phi}^{-\frac{3}{8}} \hat{\phi}' + s\hat{\phi}^\frac{3}{8} \hat{\alpha}' \leq 3T \hat{\phi}^\frac{3}{8} + 8T(1 + \gamma(\lambda)) s\hat{\phi}^\frac{3}{8} \)). This can be written as

\[
\int_Q e^{2s\hat{\alpha}} s^\frac{3}{4} \hat{\phi}^\frac{3}{4} |\nabla q|^2 \, dx \, dt \leq c(\lambda) \left( \int_Q e^{2s\hat{\alpha}} s^\frac{3}{4} \hat{\phi}^\frac{3}{4} |g|^2 \, dx \, dt + \int_Q e^{2s\hat{\alpha}} s^\frac{11}{4} \hat{\phi}^{\frac{3}{2}} |z|^2 \, dx \, dt \right).
\]

Taking (3.60) through (3.62) together and making $s$ sufficiently large, we obtain that

\[
\int_0^T e^{2s\hat{\alpha}} s^\frac{3}{4} \hat{\phi}^\frac{3}{4} |q|^2_{H^\frac{1}{2}(\partial \Omega)} \, dt \leq c(\lambda) \left( \int_{Q_{t_1}} e^{2s\hat{\alpha}} s^\frac{3}{2} \hat{\phi}^\frac{3}{2} q^2 \, dx \, dt + \int_Q e^{2s\hat{\alpha}} s^\frac{1}{2} \hat{\phi}^\alpha |z|^2 \, dx \, dt + \int_Q e^{2s\hat{\alpha}} s^{-\frac{1}{2}} \hat{\phi}^{-\frac{3}{4}} (\text{div} \ g)^2 \, dx \, dt \right)
\]

for $\lambda > \lambda_0$ and $s > s_0(\lambda)$. 27
Inserting (3.63) into inequality (3.59) we obtain (3.56), and Lemma 3.2 is proved.

Finally, coupling estimates (3.4) and (3.56), one obtains inequality (3.3), and the proof of Theorem 3.1 is finished.

4 Global exact null controllability for the linearized Navier–Stokes equations with Navier boundary conditions

We consider the linearization of the controlled equations (2.1) (with $ν = 1$) around $y_e$:

\[
\begin{align*}
\frac{∂y}{∂t} - Δy + (y_e · ∇)y + (y · ∇)y_e + ∇p &= f + χ_ωu & \text{in } Q, \\
div y &= 0 & \text{in } Q, \\
y · N &= 0 & \text{on } Σ, \\
\sum_{i,j=1}^3 \left( \frac{∂y_i}{∂x_j} + \frac{∂y_j}{∂x_i} \right) N_i T_j &= 0 & \text{on } Σ, \\
\text{for all tangential vector fields } T = (T_1, T_2, T_3) & \text{ on } ∂Ω, \\
y(·, 0) &= y_0 & \text{in } Ω,
\end{align*}
\]

\[(4.1)\]

where $y_e$ is an equilibrium solution of (2.1) and (2.2) (that is, it satisfies (2.3)). We also associate conditions (2.2) with (4.1).

One can show in the standard way (by using the Galerkin method) that, if $y_e \in (W^{1,∞}(Ω))^3$, $f \in (L^2(Q))^3$, $u \in (L^2(Ω))^3$ and $y_0 \in V$, the boundary and initial value problem (4.1), (2.2) has a unique solution $(y, p) \in (H^{2,1}(Q))^3 × L^2(0, T; H^1(Ω))$ ($p$ is unique up to a constant). Moreover, this solution satisfies the following estimate:

\[
\begin{align*}
|y|_{(H^{2,1}(Q))^3} + |∇p|_{(L^2(Q))^3} &\leq c(|y_0|_V + |f|_{(L^2(Q))^3} + |u|_{(L^2(Q))^3}).
\end{align*}
\]

(4.2)

(We refer the reader to Theorem 4.1 in [13], Theorem 1 in [14], Theorem 2 in [15], and Theorem 1 in [6] for related results.)

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To formulate and prove the required result of global controllability for the linear equations (4.1), we need the following version of function $\alpha$, which is no longer $+\infty$ at $t = 0$:

$$\beta(x, t) = e^{\lambda \psi(x)} - e^{2\lambda |\psi|_{C(\Omega)}} \frac{1}{(\theta(t)(T - t))^8},$$

where $\lambda > 0$ and $\theta$ is an increasing $C^\infty$ function such that $\theta(0) > 0$ and $\theta(t) = t$ for $t \in [T/2, T]$. The restriction of $\beta$ on $\partial \Omega$ is

$$\tilde{\beta}(t) = 1 - e^{2\lambda |\psi|_{C(\Omega)}} \frac{1}{(\theta(t)(T - t))^8}.$$ 

We shall also use some weighted $L^2$ spaces as the space $L^2(Q, (T - t)^{-8}e^{-2s\tilde{\beta}})$ of all (equivalence classes of) measurable functions $f : Q \to IR$ for which $(T - t)^{-4}e^{-s\tilde{\beta}}f \in L^2(Q)$, that is,

$$\int_Q \frac{1}{(T - t)^8} e^{-2s\tilde{\beta}}|f|^2 dx \, dt < \infty.$$ 

Now we can state the global controllability result for equations (4.1), (2.2).

**Theorem 4.1.** Let $\Omega$ and $\omega$ be as in the statement of Theorem 2.1 (or Theorem 3.1) and let $y_e \in (W^{2,4}(\Omega))^3$. Then there are $s > 0$, $\lambda > 0$ and $\delta_1 \in (1/2, 1)$ such that for any $f \in (L^2(Q, (T - t)^{-8}e^{-2s\tilde{\beta}}))^3$, $y_0 \in V$ and $\delta, \delta' \in (1/2, \delta_1]$ with $\delta' < \delta$, there exists $(u, y, p) \in (L^2(Q))^3 \times (H^{2,1}(Q))^3 \times L^2(0, T; H^1(\Omega))$ which satisfies equations (4.1), (2.2) and the final condition

$$y(x, T) = 0 \text{ a.e. } x \in \Omega,$$

and which has the following decay at $t = T$:

$$u, y \in (L^2(Q, e^{-2s\delta'\tilde{\beta}}))^3, e^{-s\delta'\tilde{\beta}}y \in (H^{2,1}(Q))^3.$$ 

**Proof.** We shall obtain a solution to the global controllability problem for equations (4.1) and (2.2) as the limit of an approximation process, constructed with the aid of a family of appropriate optimal control problems for
system (4.1). To derive the needed estimates for the solutions of the optimal control problems, we shall use both Pontryagin’s maximum principle and the Carleman estimate (3.3) for the solutions of the adjoint equations of (4.1), but the Carleman–type estimate represents the “center of mass” of the entire proof.

To formulate the optimal control problems, we need an approximation of the function \( \tilde{\beta} \) which should take a finite value at \( t = T \), too. For \( \varepsilon > 0 \) we define \( \tilde{\beta}_\varepsilon(t) \) as

\[
\tilde{\beta}_\varepsilon(t) = \frac{1 - e^{2s\theta(t)(T - t + \varepsilon)}}{\theta(t)(T - t + \varepsilon)^8}.
\]

Now for any \( \varepsilon > 0 \), we consider a corresponding optimal control problem:

Minimize

\[
(P_\varepsilon) \quad \frac{1}{2} \int_Q e^{-2s\tilde{\beta}_\varepsilon} |u|^2 \, dx \, dt + \frac{1}{2} \int_Q e^{-2s\tilde{\beta}_\varepsilon} |y|^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_\Omega |y(x, T)|^2 \, dx
\]

over all \( u \in (L^2(Q))^3 \), where \( y \) satisfies (4.1) and (2.2).

Here \( s \) and \( \lambda \) are chosen as in Theorem 3.1. Problem \( (P_\varepsilon) \) has a solution \( (u_\varepsilon, y_\varepsilon, p_\varepsilon) \) for any \( \varepsilon > 0 \), and by the convexity of the functional to minimize and the linear character of equations (4.1), this solution is even unique.

Now we shall show that \( (u_\varepsilon, y_\varepsilon, p_\varepsilon) \) really converges (on a subsequence of \( \{\varepsilon\} \)) in a certain topology as \( \varepsilon \to 0 \). The limit \( (u, y, p) \) will be proved to be a solution of the controllability problem for equations (4.1), thanks to the special form of the functional to minimize in problem \( (P_\varepsilon) \). To have convergence we need suitable estimates for \( (u_\varepsilon, y_\varepsilon, p_\varepsilon) \). The main effort will be done to obtain an \( L^2 \) estimate for \( u_\varepsilon \). The first step on this way is the use of the Pontryagin maximum principle. This states that \( (u_\varepsilon, y_\varepsilon, p_\varepsilon) \) satisfies the following necessary conditions (for optimality):

\[
(4.3) \quad u_\varepsilon = \chi_\omega e^{2s\tilde{\beta}_\varepsilon} z_\varepsilon \ \text{a.e. in } Q,
\]

where \( z_\varepsilon \) (together with a scalar function \( q_\varepsilon \)) is a solution of the (backward) adjoint equations of (4.1):
\[
\frac{\partial z_\varepsilon}{\partial t} + \Delta z_\varepsilon + (y_\varepsilon \cdot \nabla) z_\varepsilon - z_\varepsilon \cdot (\nabla y_\varepsilon) + \nabla q_\varepsilon = h_\varepsilon = e^{-2s\beta} y_\varepsilon \quad \text{in } Q,
\]
\[
\text{div } z_\varepsilon = 0 \quad \text{in } Q,
\]
\[
z_\varepsilon \cdot N = 0 \quad \text{on } \Sigma,
\]
\[
(4.4) \quad \sum_{i,j=1}^{3} \left( \frac{\partial z_{\varepsilon i}}{\partial x_j} + \frac{\partial z_{\varepsilon j}}{\partial x_i} \right) N_i T_j = 0 \quad \text{on } \Sigma,
\]
for all tangential vector fields \( T = (T_1, T_2, T_3) \) on \( \partial \Omega \),
\[
z_\varepsilon(\cdot, T) = -\frac{1}{\varepsilon} y_\varepsilon(\cdot, T) \quad \text{in } \Omega,
\]
where by the product \( z_\varepsilon \cdot (\nabla y_\varepsilon) \) we mean the field of the components
\[
(z_\varepsilon \cdot (\nabla y_\varepsilon))_i = \sum_{j=1}^{3} z_{\varepsilon j} \frac{\partial y_{\varepsilon j}}{\partial x_i}.
\]

Starting from the formula
\[
y_\varepsilon(\cdot, T), z_\varepsilon(\cdot, T))_{(L^2(\Omega))^3} - (y_\varepsilon(\cdot, 0), z_\varepsilon(\cdot, 0))_{(L^2(\Omega))^3}
\]
\[
= \int_{0}^{T} \frac{d}{dt} (y_\varepsilon(\cdot, t), z_\varepsilon(\cdot, t))_{(L^2(\Omega))^3} dt,
\]
after differentiating in the right–hand side, we use (4.1), (4.4) and (4.3) to obtain
\[
\int_{Q_\omega} e^{2s\beta} \left| z_\varepsilon \right|^2 dx \, dt + \int_{Q} e^{-2s\beta} \left| y_\varepsilon \right|^2 dx \, dt + \frac{1}{\varepsilon} \int_{\Omega} |y_\varepsilon(x, T)|^2 dx
\]
\[
= -\int_{Q} f \cdot z_\varepsilon dx \, dt - \int_{\Omega} y_0(x) \cdot z_\varepsilon(x, 0) dx
\]
\[
\leq \left( \int_{Q} e^{-2s\beta} \left( \frac{1}{(T-t)^8} |f|^2 \right) dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q} e^{2s\beta} (T-t)^8 |z_\varepsilon|^2 dx \, dt \right)^{\frac{1}{2}}
\]
\[
+ |y_0|_{(L^2(\Omega))^3} \left( \int_{\Omega} |z_\varepsilon(x, 0)|^2 dx \right)^{\frac{1}{2}}.
\]
Inequality (4.5) is the key of the present approach. Since by (4.3)
\[
\int_{Q_\omega} e^{2s\beta} |z_\varepsilon|^2 dx \, dt = \int_{Q} e^{-2s\beta} |u_\varepsilon|^2 dx \, dt,
\]
(4.5) indicates us that the desired estimate for \( u_\varepsilon \) will follow from the corresponding one for \( z_\varepsilon \).
Looking at the integrals containing $|z_\varepsilon|^2$ in both sides of (4.5), we immediately see that, to be able to get the estimate for $z_\varepsilon$, we should estimate
\[
\int_Q e^{2s\beta}(T-t)^8|z_\varepsilon|^2 \, dx \, dt \text{ and } \int_\Omega |z_\varepsilon(x,0)|^2 \, dx \text{ by } \int_{Q_\varepsilon} e^{2s\beta}|z_\varepsilon|^2 \, dx \, dt.
\]
Such estimates are usually called observability inequalities. Apparently, an observability inequality for (4.4) could be derived from the Carleman inequality (3.3) we have established for the solutions of equations (3.1), because equations (4.4) are of the form (3.1), where
\[
g = -(y_\varepsilon \cdot \nabla)z_\varepsilon + z_\varepsilon \cdot (\nabla y_\varepsilon) + e^{-2s\beta_\varepsilon} y_\varepsilon.
\]
So, we have to put this particular $g$ into the right-hand side of inequality (3.3) and next to estimate the obtained new terms by integrals containing $|z_\varepsilon|^2$ and $|\nabla z_\varepsilon|^2$ (similar to those we already have there). Then, it remains to remove the term containing $q_\varepsilon$ and to transform the observability inequality we obtain in this way into a similar one which should contain the weight functions $e^{2s\beta}$ and $e^{2s\beta_\varepsilon}$ instead of $e^{2s\alpha}$ and $e^{2s\delta_\varepsilon}$.

During the derivation of the observability inequality for (4.4), for simplicity, we shall omit $\varepsilon$ subscript after $y, z, q,$ and $h$. For $g$ written before, we have
\[
\text{div } g = -\sum_{i,j=1}^3 \frac{\partial y_{ei}}{\partial x_j} \frac{\partial z_{j}}{\partial x_i} + \nabla z \cdot \nabla y_\varepsilon + z \cdot \Delta y_\varepsilon
\]
(mind that $\text{div } y = 0$). Since $y_\varepsilon \in (W^{4,4}(\Omega))^3$, by the Sobolev imbedding theorem we also have $y_\varepsilon \in (W^{1,\infty}(\Omega))^3$, so
\[
|g| \leq c (|\nabla z| + |z|) + e^{-2s\beta_\varepsilon}|y| \quad \text{a.e. in } Q,
\]
(4.6)
\[
|\text{div } g| \leq c |\nabla z| + |\Delta y_\varepsilon||z| \quad \text{a.e. in } Q.
\]
(4.7)

Now let us estimate the terms containing $g$ in (3.3), where $g$ is that before. First, by (4.6) it is easy to see that
\[
\int_Q e^{2s\alpha}|g|^2 \, dx \, dt + \int_Q e^{2s\alpha} s^{-\frac{3}{4}} \phi^3 |g|^2 \, dx \, dt
\]
\[
\leq c(\lambda) \left( \int_Q e^{2s\alpha} s^{-\frac{3}{4}} \phi^3 (|\nabla z|^2 + |z|^2) \, dx \, dt + \int_Q e^{2s\delta_\varepsilon} |h|^2 \, dx \, dt \right)
\]
for $0 < \delta \leq \delta(\lambda) < 1.
(because $\hat{\theta} \leq \alpha$, $\hat{\varphi} \leq \varphi$, $e^{2s\alpha} < e^{2s\theta}$, and $s^3\hat{\varphi}^3 e^{2s\alpha} < ce^{2s\theta}$ for some $c > 0$ depending on $\delta$). Here $\delta(\lambda) = (e^{2\lambda|\psi|_{C(\Omega)}} - e^{\lambda|\psi|_{C(\Omega)}})/ (e^{2\lambda|\psi|_{C(\Omega)}} - 1)$. Note that $\delta(\lambda) > 1/2$ when $\lambda > \log 2/|\psi|_{C(\Omega)}$.

By (4.7) we can write

$$
\int_Q e^{2s\alpha} s^2 \hat{\varphi}^2 \text{div} g^2 \, dx \, dt \\
\leq c \int_Q e^{2s\alpha} s^2 \hat{\varphi}^2 \text{div} z^2 \, dx \, dt + \int_Q e^{2s\alpha} s^2 \hat{\varphi}^2 \Delta y_e \, dx \, dt.
$$

(4.9)

Let us see the second integral in the right-hand side. Using Schwarz’s inequality and taking into account the inclusions $y_e \in (W^{2,4}(\Omega))^3$ and $(H^1(\Omega))^3 \subset (L^6(\Omega))^3$ (the latter derived from the Sobolev imbedding theorem), we have

$$
\int_Q e^{2s\alpha} s^2 \hat{\varphi}^2 \Delta y_e \, dx \, dt \\
\leq \int_0^T \hat{\varphi}^2 |\Delta y_e|_{L^4(\Omega)}^2 |e^{s\alpha} z|_{L^4(\Omega)}^2 \, dt \\
\leq c_1 \int_0^T \hat{\varphi}^2 |e^{s\alpha} z|_{L^4(\Omega)}^2 \, dt \\
\leq c_2 \int_0^T \hat{\varphi}^2 |e^{s\alpha} z|_{H^1(\Omega)}^3 |e^{s\alpha} z|_{L^2(\Omega)}^3 \, dt \\
\leq c_3 \int_0^T \hat{\varphi}^2 |e^{s\alpha} z|_{L^6(\Omega)}^2 |e^{s\alpha} z|_{L^2(\Omega)}^3 \, dt \\
\leq c_4 \int_0^T \hat{\varphi}^2 (|e^{s\alpha} z|_{L^2(\Omega)}^2 + |\nabla (e^{s\alpha} z)|_{L^2(\Omega)}^2) \, dt \\
\leq c_5(\lambda) \int_Q e^{2s\alpha} \hat{\varphi}^2 (|\nabla z|^2 + s^2 \hat{\varphi}^2 |z|^2) \, dx \, dt.
$$

(4.10)

(The third and the fifth inequality in (4.10) follow by Hölder’s and Young’s inequalities, respectively.) Taking (4.9) and (4.10) together, we obtain

$$
\int_Q e^{2s\alpha} s^2 \hat{\varphi}^2 \text{div} g^2 \, dx \, dt \\
\leq c(\lambda) \int_Q e^{2s\alpha} (s^2 \hat{\varphi}^2 |\nabla z|^2 + s^2 \hat{\varphi}^2 |z|^2) \, dx \, dt.
$$

(4.11)

Since the powers of $s$ and $\varphi$ before $|z|^2$ and $|\nabla z|^2$ are greater in the left-hand side of (3.3) than in the right-hand side of (4.8) and (4.11), combining (3.3), taken for a fixed $\lambda = \lambda_1 > \max(\lambda_0, \log 2/|\psi|_{C(\Omega)})$, with (4.8) and (4.11), we obtain the following Carleman–type inequality for the solutions $(z, q)$ of equations (4.4):

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\[
\int_Q e^{2s\alpha} \left( \frac{1}{s\varphi} \left( \left| \frac{\partial z}{\partial t} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + s\varphi |\nabla z|^2 + s^3 \varphi^3 |z|^2 \right) dx dt \\
\leq c \left( \int_{Q_{e^2s\alpha}} e^{2s^3\varphi^3} |z|^2 dx dt + \int_{Q_{e^{s\alpha}}} e^{2s^3\varphi^3} \tilde{\varphi}^{1/4} q^2 dx dt + \int_Q e^{2s\alpha} |h|^2 dx dt \right) \\
\text{for } s > s_1 = s_0(\lambda_1) \text{ and } 0 < \delta \leq \delta_1 = \delta(\lambda_1) < 1.
\]

Notice that \(\delta_1 > 1/2\) (because \(\lambda_1 > \log 2/|\psi|_{C(\Omega)}\)).

Now let us deal with the removal of the term containing \(q\) in (4.12). At a first glance, this would seem to follow from an interior estimate for \(\nabla q\), obtained by using equations (4.4). If, to get such an estimate, one would work with the \((L^2(\omega_1))^3\) norm of \(\nabla q\), then, because of the term \((y e \cdot \nabla)z\) in (4.4), this would produce an integral as \(\int_Q e^{2s\alpha} \nabla z^2 dx dt\), for some \(s \in (0, 1)\), which cannot be absorbed by the integral containing \(|\nabla z|^2\) we already have in the left–hand side of (4.12). Thus, it is reasonable to take \(\nabla q\) in the \(((H^1_0(\omega_1))^3)'\) norm. But doing so, we should have difficulties to estimate \(\partial z/\partial t\) in the \(((H^1_0(\omega_1))^3)'\) norm, because the norms of \(\nabla q\) in \(((H^1_0(\omega_1))^3)\)' and \(V'\) are generally different. So, to be able to remove \(q\) in (4.12), we are forced to use an elliptic variant of equations (4.4). This indirect way was followed by Imanuvilov in [9] and [10]. For reader’s convenience, we shall develop the arguments in almost all their details.

For \(t \in [T/2, T]\), we set

\[
\bar{z}(x, t) = \int_{T/2}^{t} z(x, \tau) d\tau, \quad \bar{q}(x, t) = \int_{T/2}^{t} q(x, \tau) d\tau, \quad \bar{h}(x, t) = \int_{T/2}^{t} h(x, \tau) d\tau.
\]

Integrating equations (4.4) from \(T/2\) to \(t\), we easily see that \(\bar{z}, \bar{q}\) and \(\bar{h}\) satisfy the following equations:

\[
\begin{align*}
\Delta \bar{z} + \nabla \bar{q} &= \bar{h} - (y e \cdot \nabla)\bar{z} + \bar{z} \cdot (\nabla y e) - z + z \left( \frac{T}{2} \right) & \text{in } \Omega \times \left( \frac{T}{2}, T \right), \\
\div \bar{z} &= 0 & \text{in } \Omega \times \left( \frac{T}{2}, T \right).
\end{align*}
\]

Since \(z = \partial z/\partial t\), \(\bar{z}, \bar{q}\) and \(\bar{h}\) also satisfy
\[
\frac{\partial z}{\partial t} + \Delta z + (y_e \cdot \nabla) z - z \cdot (\nabla y_e) + \nabla \varrho = \overline{h} + z \left( \cdot, \frac{T}{2} \right) \quad \text{in } \Omega \times \left( \frac{T}{2}, T \right),
\]
\[
\text{div } z = 0 \quad \text{in } \Omega \times \left( \frac{T}{2}, T \right),
\]
(4.14) \quad z \cdot N = 0 \quad \text{on } \partial \Omega \times \left( \frac{T}{2}, T \right),
\]
\[
\sum_{i,j=1}^3 \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) N_i T_j = 0 \quad \text{on } \partial \Omega \times \left( \frac{T}{2}, T \right),
\]

for all tangential vector fields \( T = (T_1, T_2, T_3) \) on \( \partial \Omega \).

(Let us notice that equations (4.14) are identical with equations (4.4), except the different right–hand sides \( h + z(\cdot, T/2) \) (in (4.14)) and \( h \) (in the other). So, the Carleman inequality (4.12), with \( h \) replaced by \( \overline{h} + z(\cdot, T/2) \), is also satisfied by \( z \) and \( q \).)

We can choose \( q \) in (4.4) such that
\[
\int_{\omega_1} q(x,t) dx = 0 \quad \text{for all } t \in [0,T]
\]
(because one can replace \( q \) in (4.4) by \( \tilde{q} = q - (\text{meas } \omega_1)^{-1} \int_{\omega_1} q \, dx \)). This implies
\[
\int_{\omega_1} \tilde{q}(x,t) dx = 0 \quad \text{for } t \in [0,T],
\]
and so, by Proposition 1.2 in [17], we have
\[
(4.15) \quad \int_{\omega_1} \int_{\frac{T}{2}}^T e^{2s\alpha} (s\tilde{\phi})^{\frac{11}{12}} \tilde{\varrho}^2 \, dx \, dt \leq c_1 \int_{\frac{T}{2}}^T \int_{\omega_1} e^{2s\delta} \tilde{\varrho}^2 \, dx \, dt
\]
\[
\leq c_2 \int_{\frac{T}{2}}^T e^{2s\delta} |\nabla \tilde{\varrho}|^2_{(H^1_0(\omega_1))'} \, dt \quad \text{for } 0 < \delta \leq \delta_1 < 1,
\]
where \( \delta_1 \) is that in (4.12) (see (4.8), too).

To estimate the \( (H^1_0(\omega_1))^3' \) norm of \( \nabla \varrho \) in (4.15), we shall use equations (4.13), viewed as an elliptic system. Let \((\overline{z}_1, \overline{q}_1)\) be the solution of the following steady state Stokes equations with Dirichlet boundary condition:
\[
(4.16) \quad \Delta \overline{z}_1 + \nabla \overline{q}_1 = \overline{h} - (y_e \cdot \nabla) \overline{z} + \overline{z} \cdot (\nabla y_e) - z + z \left( \cdot, \frac{T}{2} \right) \quad \text{in } \omega,
\]
\[
\text{div } \overline{z}_1 = 0 \quad \text{in } \omega,
\]
\[
z_1 = 0 \quad \text{on } \partial \omega.
\]
(Of course, \(z_1\) and \(q_1\) depend on \(t\), too, because of the fields \(\overline{h}, \overline{z}\) and \(z\) in the right–hand side of (4.16).) Subtracting equations (4.13) and (4.16), we immediately see that \(z_2 = \overline{z} - z_1\) and \(q_2 = \overline{q} - q_1\) satisfy the Stokes system

\[
\begin{align*}
\Delta z_2 + \nabla q_2 &= 0 \quad \text{in } \omega, \\
\text{div } z_2 &= 0 \quad \text{in } \omega.
\end{align*}
\]

From equations (4.16), we have

\[
|\nabla q_1|_{((H_0^1(\omega))^3)'} \leq |\overline{h}|_{(L^2(\omega))^3} + |z_1|_{(H^1(\omega))}^3 + |z|_{((H^1(\omega))^3)'} + \left|z \left(\cdot, \frac{T}{2}\right)\right|_{((H_0^1(\omega))^3)'}.
\]

After using the well–known estimate for the weak solutions of the steady state Stokes equations with null boundary conditions, the above inequality becomes

\[
|\nabla q_1|_{((H_0^1(\omega))^3)'} \leq c \left(|\overline{h}|_{(L^2(\omega))^3} + |z_1|_{(L^2(\omega))}^3ight.
\]

\[
+ \left| - (y_e \cdot \nabla)\overline{z} + \overline{z} \cdot (\nabla y_e) \right|_{((H_0^1(\omega))^3)'} + \left|z \left(\cdot, \frac{T}{2}\right)\right|_{((H_0^1(\omega))^3)'}.
\]

To estimate the \(((H_0^1(\omega))^3)')\ norm of \(- (y_e \cdot \nabla)\overline{z} + \overline{z} \cdot (\nabla y_e)\) in (4.18), we shall use a well–known inequality for the trilinear form \(b\) defined as

\[
b(u, v, w) = \int_\Omega ((u \cdot \nabla)v) \cdot w \, dx = \sum_{i,j=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.
\]

We have (see [5] or [16])

\[
|b(u, v, w)| \leq c|u|_{(H^{m_1}(\Omega))^3}|v|_{(H^{m_2+1}(\Omega))^3}|w|_{(H^{m_3}(\Omega))^3}
\]

for \(m_1, m_2, m_3 \geq 0\) with \(m_1 + m_2 + m_3 > 3/2\) or \(m_1 + m_2 + m_3 = 3/2\) and at least two \(m_i\) are nonzero, and for \((u, v, w) \in (H^{m_1}(\Omega))^3 \times (H^{m_2+1}(\Omega))^3 \times (H^{m_3}(\Omega))^3\). Moreover, \(b\) has also the following symmetry property:

\[
b(u, v, w) = -b(u, w, v) \quad \text{for } u \in V \text{ and } v, w \in (H^1(\Omega))^3,
\]

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which comes from the fact that
\[ b(u, v, v) = 0 \text{ for } u \in V \text{ and } v \in (H^1(\Omega))^3. \]

Denoting by \( \tilde{z} \) the field defined as \( z \) in \( \omega \) and 0 in \( \Omega \setminus \omega \), we have for \( 3/2 < \gamma \leq 2 \),
\[
\left| \int_{\Omega} (-y_e \cdot \nabla) z + z \cdot (\nabla y_e) \cdot \chi \, dx \right|
= | - b(y_e, z, \chi) + b(\chi, y_e, z) | \leq | b(y_e, \chi, \tilde{z}) | + | b(\chi, y_e, \tilde{z}) |
\leq c | y_e | (H^\gamma(\Omega))^3 | z | (L^2(\omega))^3 | \chi | (H^1(\omega))^3 \text{ a.e. on } (0, T), \text{ for all } \chi \in (H^\gamma(\omega))^3.
\]

Thus we have obtained
\[
| - (y_e \cdot \nabla) z + z \cdot (\nabla y_e) | (H^\gamma(\omega))^3 | z | (L^2(\omega))^3 | \chi | (H^1(\omega))^3 \text{ a.e. on } (0, T),
\]
because \( y_e \in (W^{2,4}(\Omega))^3 \).

Inserting (4.19) into inequality (4.18), we have
\[
| \nabla q_1 | (H^\gamma(\omega))^3 | q_1 | (H^\gamma(\omega))^3 | \chi | (L^2(\omega))^3 + | z | (L^2(\omega))^3 + \left| \frac{z}{\gamma} \right| (L^2(\omega))^3 \right) \right).
\]

Now applying the divergence operator to both sides of equations (4.17), we obtain that
\[ \Delta q_2 = 0 \text{ in } \omega. \]
Then applying the Laplace operator to (4.17), since \( \overline{q}_2 \) is harmonic, we have
\[ \Delta^2 \overline{z}_2 = 0 \text{ in } \omega. \]

A well-known interior (regularity) estimate for biharmonic functions gives
\[
| \overline{z}_2 | (H^2(\omega))^3 \leq c | \overline{z}_2 | (L^2(\omega))^3 \leq c (| \overline{z} | (L^2(\omega))^3 + | \overline{z}_1 | (L^2(\omega))^3).\]

As before, the estimate for the weak solution of (4.16) produces
\[
| \overline{z}_1 | (H^1(\omega))^3 \leq c \left( | \overline{z} | (L^2(\omega))^3 + | \overline{z}_1 | (L^2(\omega))^3 + \left| \frac{z}{\gamma} \right| (L^2(\omega))^3 \right).
\]
(Here we have also used inequality (4.19).) From equations (4.17), we have

\[ |\nabla q_2|_{(H^3_0(\omega))} \leq |\tau_2|_{(H^1(\omega))}. \]

This together with (4.21) and (4.22) yields

\[
\nabla q_2 \big|_{(H^3_0(\omega))} \\
\leq c \left( |T|_{(L^3(\omega))}^3 + |\tau|_{(L^3(\omega))}^3 + |z|_{(L^3(\omega))}^3 + \left| \frac{z}{\cdot}, \frac{T}{2} \right|_{(L^3(\omega))}^3 \right).
\]

Thus, by (4.20) and (4.23), we have

\[
\nabla q \big|_{(H^3_0(\omega))} \\
\leq c \left( |T|_{(L^3(\omega))}^3 + |\tau|_{(L^3(\omega))}^3 + |z|_{(L^3(\omega))}^3 + \left| \frac{z}{\cdot}, \frac{T}{2} \right|_{(L^3(\omega))}^3 \right).
\]

Since the function \( \hat{\alpha} \) decreases on \([T/2, T]\), it is easy to see that

\[
\int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} |T|^2 dx dt \leq \frac{T}{2} \int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} |h|^2 dx dt,
\]

\[
\int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} |\tau|^2 dx dt \leq \frac{T}{2} \int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} |z|^2 dx dt.
\]

Besides, as \( \hat{\alpha} \) attains its maximum at \( t = T/2 \), we have

\[
\int_T^{T/2} e^{2s\hat{\alpha}} \left| z \left( \cdot, \frac{T}{2} \right) \right|_{(L^3(\omega))}^2 dt \leq \frac{T}{2} e^{2s\hat{\alpha}(T/2)} \left| z \left( \cdot, \frac{T}{2} \right) \right|_{(L^3(\omega))}^2.
\]

Putting inequalities (4.15) and (4.24) through (4.26) together, we obtain

\[
\int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} (s\hat{\varphi})_{11}^2 \frac{q^2}{q} dx dt \\
\leq c \left( \int_T^{T/2} \int_\omega e^{2s\hat{\alpha}} \left( |h|^2 + |z|^2 \right) dx dt + \left| z \left( \cdot, \frac{T}{2} \right) \right|_{(L^3(\omega))}^2 \right) \text{ for } 0 < \delta \leq \delta_1.
\]

Repeating the considerations before on the interval \([0, T/2]\), we obtain the above inequality on this interval, too. So, we can write

\[
\int_{Q_{\omega_1}} e^{2s\hat{\alpha}} (s\hat{\varphi})_{11}^2 \frac{q^2}{q} dx dt \\
\leq c \left( \int_{Q_{\omega_1}} e^{2s\hat{\alpha}} |z|^2 dx dt + \int_{Q_{\omega_1}} e^{2s\hat{\alpha}} |h|^2 dx dt + \left| z \left( \cdot, \frac{T}{2} \right) \right|_{(L^3(\omega))}^2 \right) \text{ for } 0 < \delta \leq \delta_1.
\]
Now taking the Carleman inequality (4.12) for the solution \((\pi, q)\) of (4.14) (so, in (4.12), \(h = e^{-2s\delta_\beta} y\) is replaced by \(\bar{h} + z(\cdot, T/2)\)) together with (4.27) and (4.25) (where \(\omega\) is replaced by \(\Omega\)), we obtain

\[
\int_Q e^{2s\alpha} \frac{1}{\varphi} |z|^2 \, dx \, dt
\]

(4.28)

\[
\leq c(s) \left( \int_{Q_\omega} e^{2s\delta_\tilde{\beta}} |z|^2 \, dx \, dt + \int_Q e^{-2s\delta_\beta} |y|^2 \, dx \, dt + \left| z \left( \cdot, \frac{T}{2} \right) \right|^2_{(L^2(\Omega))^3} \right)
\]

for \(s > s_1\) and \(0 < \delta \leq \delta_1\).

(Here we have also used the fact that \(\tilde{\alpha} \leq \tilde{\beta} < \tilde{\beta}_\varepsilon\).) Inequality (4.28) would look like an "observability" inequality whether it would not contain the \((L^2(\Omega))^3\) norm of \(z(\cdot, T/2)\) in the right-hand side. To remove that term, we will argue by contradiction. This reasoning requires an estimate for the \((H^1(\Omega))^3\) norm of \(z(\cdot, T/2)\) by its \((L^2(\Omega))^3\) norm. Let us get it in the following.

Let \(\rho \in C^\infty([0, T])\) such that \(\rho = 1\) on \([0, T/2]\) and \(\rho = 0\) on \([3T/4, T]\). We set \(\tilde{z} = \rho z\) and \(\tilde{q} = \rho q\). It is easy to check that \(\tilde{z}\) and \(\tilde{q}\) satisfy equations (4.4) (just like \(z\) and \(q\)), where in the right-hand side, besides \(e^{-2s\delta_\beta_\varepsilon} \rho y\), the term \(\rho'z\) is found, and whose final condition is \(\tilde{z}(\cdot, T) = 0\) (in fact, \(\tilde{z}(\cdot, t) = 0\) for all \(t \in [3T/4, T]\)). Using the trace theorem (see Theorem 2.1 in [12], Vol. II) and an estimate as (4.2) for the backward equations (4.4), we obtain (mind that \(\tilde{\alpha} \leq \tilde{\beta}_\varepsilon\))

\[
\left| z \left( \cdot, \frac{T}{2} \right) \right|^2_{(H^1(\Omega))^3} = \left| \tilde{z} \left( \cdot, \frac{T}{2} \right) \right|^2_{(H^1(\Omega))^3} \leq c_1 |\tilde{z}|_{(H^{2,1}(\Omega \times [\frac{T}{2}, \frac{3T}{4}]))^3}
\]

\[
\leq c_2 \left( |e^{-2s\delta_\beta_\varepsilon} \rho y|_{(L^2(\Omega \times [\frac{T}{2}, \frac{3T}{4}]))^3} + |\rho'z|_{(L^2(\Omega \times [\frac{T}{2}, \frac{3T}{4}]))^3} \right)
\]

\[
\leq c_3(s) \left( \int_{\frac{3T}{4}}^{\frac{T}{2}} \int_\Omega e^{-2s\delta_\beta_\varepsilon} |y|^2 \, dx \, dt + \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_\Omega e^{2s\alpha} \frac{1}{\varphi} |z|^2 \, dx \, dt \right).
\]

Then using inequality (4.28), too, we have

\[
\left| z \left( \cdot, \frac{T}{2} \right) \right|^2_{(H^1(\Omega))^3}
\]

(4.29)

\[
\leq c(s) \left( \int_{Q_\omega} e^{2s\delta_\tilde{\beta}} |z|^2 \, dx \, dt + \int_Q e^{-2s\delta_\beta} |y|^2 \, dx \, dt + \left| z \left( \cdot, \frac{T}{2} \right) \right|^2_{(L^2(\Omega))^3} \right)
\]

for \(s > s_1\) and \(0 < \delta \leq \delta_1\).
Now let us suppose (by contradiction) that the inequality obtained from (4.28) by removing the square of the \((L^2(\Omega))^3\) norm of \(z(\cdot, T/2)\) would not be true. Then for any positive integer \(k\), one could find \(z_k, q_k\) and \(y_k\) satisfying equations (4.4), such that

\[
(4.30) \quad \int_Q e^{2s\alpha} \frac{1}{\varphi} |z_k|^2 \, dx \, dt > k \left( \int_{Q(\omega)} e^{2s\delta \hat{\kappa}} |z_k|^2 \, dx \, dt + \int_{Q} e^{-2s\delta \hat{\kappa}} |y_k|^2 \, dx \, dt \right)
\]

and (thanks to the fact that equations (4.4) are linear with respect to \(z, q\) and \(y\), and homogeneous)

\[
|z_k(\cdot, T/2)|_{(L^2(\Omega))^3} = 1.
\]

Inequality (4.30) combined with (4.28) yields

\[
\left( 1 - \frac{c(s)}{k} \right) \int_Q e^{2s\alpha} \frac{1}{\varphi} |z_k|^2 \, dx \, dt < c(s) \left| z_k(\cdot, T/2) \right|_{(L^2(\Omega))^3}^2 = c(s),
\]

where \(c(s)\) is that in (4.28). This taken together with (4.30) gives

\[
(4.31) \quad \int_{Q(\omega)} e^{2s\delta \hat{\kappa}} |z_k|^2 \, dx \, dt + \int_{Q} e^{-2s\delta \hat{\kappa}} |y_k|^2 \, dx \, dt < \frac{c(s)}{k} \frac{1}{1 - c(s)/k}.
\]

Consequently, as \(k \to 0\),

\[
y_k \to 0 \text{ in } L^2(Q, e^{-2s\delta \hat{\kappa}}) \text{ and } z_k \to 0 \text{ in } L^2(Q(\omega), e^{2s\delta \hat{\kappa}}).
\]

By inequalities (4.29) and (4.31), we have

\[
\left| z_k(\cdot, T/2) \right|_{(H^1(\Omega))^3}^2 < c(s),
\]

where \(c(s)\) is a little greater than that in (4.29) (mind that \(\left| z_k(\cdot, T/2) \right|_{(L^2(\Omega))^3}^2 = 1\)). An estimate like (4.2) for equations (4.4) together with the preceding inequality and (4.31) gives

\[
\left| z_k \right|_{(H^{2,1}(\Omega \times [0, T/2])^3) \leq c(s)}
\]

and

\[
\left| \nabla q_k \right|_{(L^2(\Omega \times [0, T/2])^3) \leq c(s)}.
\]
for some $c(s) > 0$. Replacing $q_k$ (if necessary) by $q_k - (\text{meas } \Omega)^{-1} \int_{\Omega} q_k \, dx$, we also can write

$$|q_k|_{L^2(0, \frac{T}{2}; H^1(\Omega))} \leq c(s).$$

Thus, for some $z \in (H^{2,1}(\Omega \times [0, T/2]))^3$ and $q \in L^2(0, T/2; H^1(\Omega))$, on a subsequence of $\{k\}$,

$$z_k \longrightarrow z \text{ weakly in } \left( H^{2,1}(\Omega \times [0, \frac{T}{2}]) \right)^3,$$

and

$$q_k \longrightarrow q \text{ weakly in } L^2 \left( 0, \frac{T}{2}; H^1(\Omega) \right),$$

as $k \to \infty$. Passing to the limit in (4.4) as $k \to \infty$, we obtain that $z$ and $q$ satisfy:

$$\frac{\partial z}{\partial t} + \Delta z + (y_e \cdot \nabla) z - z \cdot (\nabla y_e) + \nabla q = 0 \quad \text{in } \Omega \times \left( 0, \frac{T}{2} \right),$$
$$\text{div } z = 0 \quad \text{in } \Omega \times \left( 0, \frac{T}{2} \right),$$
$$z \cdot N = 0 \quad \text{on } \partial \Omega \times \left( 0, \frac{T}{2} \right),$$
$$\sum_{i,j=1}^{3} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right) N_i T_j = 0 \quad \text{on } \partial \Omega \times \left( 0, \frac{T}{2} \right),$$

for all tangential vector fields $T = (T_1, T_2, T_3)$ on $\partial \Omega$.

Besides,

$$z = 0 \text{ on } \omega \times \left[ 0, \frac{T}{2} \right] \text{ and } |z(\cdot, \frac{T}{2})|_{(L^2(\Omega))^3} = 1.$$

By a Carleman inequality as (4.12), where $T$ is replaced by $T/2$, $\alpha$ and $\varphi$ are suitably modified, and $h = 0$, we derive that

$$z = 0 \text{ on } \Omega \times \left[ 0, \frac{T}{2} \right].$$

But this contradicts the fact that the $(L^2(\Omega))^3$ norm of $z(\cdot, T/2)$ is 1. In this way, we have proved the first "observability" inequality for system (4.4) we need:
\[
\int_0^T \int_\Omega e^{2s\alpha} \frac{1}{\phi} |z|^2 dx \, dt \leq c(s) \left( \int_{Q_\omega} e^{2s\delta \hat{\alpha}} |z|^2 dx \, dt + \int_Q e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right)
\]
for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 < 1 \).

Now let us convert this inequality into a similar one which contains the weight functions \( e^{2s\beta} \) and \( e^{2s\delta \beta} \) instead of \( e^{2s\alpha} \) and \( e^{2s\delta \hat{\alpha}} \). The key to succeed in doing so is the following estimate for \( z \):

\[
\int_0^T \int_\Omega |z|^2 dx \, dt \leq c(s) \left( \int_{T/2}^T \int_\Omega |z|^2 dx \, dt + \int_0^{T/2} \int_\Omega e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right),
\]
for some positive \( c(s) \) (independent of \( z \)). To prove (4.33) it suffices pass from \( z \) to \( \tilde{z} = \rho z \), where \( \rho \in C^\infty([0,T]) \) and takes the values \( \rho = 1 \) on \([0,T/2]\) and \( \rho = 0 \) on \([3T/4,T]\). The new \( \tilde{z} \) (together with a new \( \tilde{\rho} = \rho q \)) satisfies equations similar to (4.4) but having \( \rho' z \) (besides \( e^{-2s\delta \hat{\beta}} \rho y \)) in the right-hand side and \( \tilde{z}(\cdot,T) = 0 \) as final condition (\( \tilde{z}(\cdot,t) = 0 \) for \( t \in [3T/4,T], \) too).

Then (4.33) follows by simply using an estimate as (4.2) for the solutions of equations of type (4.4). (We refer the reader to the appendix in [10] for more details.) Now taking into account that \( \hat{\beta}(t) = \hat{\alpha}(t) \leq \alpha(t) \) for \( T/2 \leq t \leq T \) (but also the fact that \( \hat{\beta}(t) > \hat{\alpha}(t) \) for \( 0 \leq t < T/2 \)) and using (4.32), we have for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 \),

\[
\int_{T/2}^T \int_\Omega e^{2s\delta \hat{\beta}} (T - t)^\delta |z|^2 dx \, dt \leq c_1 \int_{T/2}^T \int_\Omega e^{2s\alpha} \frac{1}{\phi} |z|^2 dx \, dt
\]
\[
\leq c_2(s) \left( \int_{Q_\omega} e^{2s\delta \hat{\beta}} |z|^2 dx \, dt + \int_Q e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right).
\]

Next using (4.33) and (4.34), we obtain for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 \),

\[
\int_0^{T/2} \int_\Omega e^{2s\delta \hat{\beta}} (T - t)^\delta |z|^2 dx \, dt \leq c_1 \int_0^{T/2} \int_\Omega |z|^2 dx \, dt
\]
\[
\leq c_2(s) \left( \int_{T/2}^T \int_\Omega |z|^2 dx \, dt + \int_0^{T/2} \int_\Omega e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right)
\]
\[
\leq c_3(s) \left( \int_{T/2}^T \int_\Omega e^{2s\hat{\beta}} (T - t)^\delta |z|^2 dx \, dt + \int_0^{T/2} \int_\Omega e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right)
\]
\[
\leq c_4(s) \left( \int_{Q_\omega} e^{2s\delta \hat{\beta}} |z|^2 dx \, dt + \int_Q e^{-2s\delta \hat{\beta}} |y|^2 dx \, dt \right).
\]
Inequalities (4.34) and (4.35) taken together give the variant of the "observability" inequality for (4.4) we need in order to estimate the first term in the right–hand side of (4.5):

\[
\int_Q e^{2s\hat{\beta}}(T - t)^8|z|^2 dx dt \leq c(s) \left( \int_Q e^{2s\hat{\beta}}|z|^2 dx dt + \int_Q e^{-2s\hat{\beta}}|y|^2 dx dt \right)
\]

for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 < 1 \).

To estimate the second term in the right–hand side of (4.5), too, we have to derive a genuine observability inequality for system (4.4), that is, an estimate for the initial state on \( \Omega \) (of the backward system) by means of the states taken on \( \omega \) at all the subsequent moments. To this aim, we scalarly multiply the first equation in (4.4) (with \( z \) instead of \( z_\varepsilon \)) by \( z \) and integrate over \( \Omega \). We obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |z|^2 dx - \frac{1}{2} \sum_{i,j=1}^3 \int_{\Omega} \left( \frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i} \right)^2 dx - b(z, y_e, z) = \int_{\Omega} e^{-2s\hat{\beta}} y \cdot z dx \text{ a.e. in } (0, T).
\]

Since \( y_e \in (W^{2,4}(\Omega))^3 \), we also have \( y_e \in (W^{1,\infty}(\Omega))^3 \), which leads to the inequality

\[
|b(z, y_e, z)| \leq c |z|^2 \text{ in } (L^2(\Omega))^3,
\]

for some positive constant \( c \) (independent of \( z \)). Now we take \( t_1 \in (0, T) \).

Next we choose \( \gamma > 0 \) such that \( -\hat{\beta}(t) \leq \gamma \) for \( 0 < t < t_1 \), whence we also have \( -\hat{\beta}(t) \leq \gamma \) for \( 0 < t < t_1 \) and \( \varepsilon > 0 \). Integrating (4.37) over \((0, t)\) with \( 0 < t < t_1 \) and then using (4.38), we obtain

\[
\int_{\Omega} |z(x, 0)|^2 dx \\
\leq \frac{1}{(T-t_1)^2} \int_0^{t_1} \int_{\Omega} e^{2s\hat{\beta}}(T - \tau)^8|z|^2 dx d\tau + e^{2s\gamma} \int_0^{t_1} \int_{\Omega} e^{-2s\hat{\beta}}|y|^2 dx d\tau \\
+ 2c \frac{1}{(T-t_1)^2} \int_0^{t_1} \int_{\Omega} e^{2s\hat{\beta}}(T - \tau)^8|z|^2 dx d\tau \\
+ \frac{1}{(T-t_1)^2} e^{2s\gamma} \int_{\Omega} e^{2s\hat{\beta}}(T - t)^8|z(x, t)|^2 dx
text{ for } t \in (0, T)
\]

(where \( c \) is the constant in (4.38)). Integrating over \((0, t_1)\) (with respect to \( t \)), we have

\[
\int_{\Omega} |z(x, 0)|^2 dx \leq c(s) \left( \int_Q e^{2s\hat{\beta}}(T - t)^8|z|^2 dx dt + \int_Q e^{-2s\hat{\beta}}|y|^2 dx dt \right).
\]
which combined with (4.36) gives the following observability inequality for system (4.4):

\[
\int_\Omega |z(x,0)|^2 dx \leq c(s) \left( \int_{Q_\omega} e^{2s\delta}\beta |z|^2 dx dt + \int_Q e^{-2s\delta}\beta |y|^2 dx dt \right)
\]

for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 \).

Now we fix the parameter \( s > s_1 \). Inserting (4.36) and (4.39) (where \( z \) and \( y \) have been replaced by \( z_\varepsilon \) and \( y_\varepsilon \)) into (4.5), we obtain the following estimate for \( z_\varepsilon \) and \( y_\varepsilon \):

\[
\int_{Q_\omega} e^{2s\delta}\beta |z_\varepsilon|^2 dx dt + \int_Q e^{-2s\delta}\beta |y_\varepsilon|^2 dx dt
\]

\[
\leq 2c \left( |y_0|^2_{L^2(\Omega)^3} + \int_Q e^{-2s\beta} \frac{1}{(T-t)^8} |f|^2 dx dt \right)
\]

(4.40)

for \( s > s_1 \) and \( 0 < \delta \leq \delta_1 \). By (4.3) this means that

\[
\int_{Q_\omega} e^{-2s\delta\beta} |u_\varepsilon|^2 dx dt + \int_Q e^{-2s\delta}\beta |y_\varepsilon|^2 dx dt
\]

\[
\leq c \left( |y_0|^2_{L^2(\Omega)^3} + \int_Q e^{-2s\beta} \frac{1}{(T-t)^8} |f|^2 dx dt \right)
\]

(4.41)

for all \( \varepsilon > 0 \).

Besides, (4.5), (4.36), (4.39) and (4.40) taken together give

\[
\int_\Omega |y_\varepsilon(x,T)|^2 dx \leq c \varepsilon
\]

(4.42)

for some positive constant \( c \) independent of \( \varepsilon \).

The boundedness of \( u_\varepsilon \) in \( (L^2(Q))^3 \) given by (4.41) (by (4.3), \( u_\varepsilon = 0 \) outside \( \omega \)) implies that of \( y_\varepsilon \) and \( \nabla p_\varepsilon \) in \( (H^{2,1}(Q))^3 \) and \( (L^2(Q))^3 \), respectively, via estimate (4.2). If we take \( q_\varepsilon = p_\varepsilon - (\text{meas}\Omega)^{-1} \int_\Omega p_\varepsilon dx \) instead of \( p_\varepsilon \), too, the new \( p_\varepsilon \) together with \( y_\varepsilon \) also satisfies equations (4.1) but, besides, it is bounded in \( L^2(0,T; H^1(\Omega)) \) (by Proposition 1.2 in [17]). Hence we infer that there exists \( (u, y, p) \in (L^2(Q))^3 \times (H^{2,1}(Q))^3 \times L^2(0,T; H^1(\Omega)) \) such that, on a subsequence of \( \{\varepsilon\} \) (also denoted by \( \{\varepsilon\} \)), we have as \( \varepsilon \to 0 \):

\[
\begin{align*}
 u_\varepsilon & \rightharpoonup u \quad \text{weakly in } (L^2(Q))^3, \\
y_\varepsilon & \rightharpoonup y \quad \text{weakly in } (H^{2,1}(Q))^3 \text{ and}
\quad \text{weakly in } L^2(0,T; (H^1(\partial\Omega))^3), \\
p_\varepsilon & \rightharpoonup p \quad \text{weakly in } L^2(0,T; H^1(\Omega)).
\end{align*}
\]
But the last two convergences before obviously give the following ones: as $\varepsilon \to 0$,

$$y_\varepsilon \rightharpoonup y \quad \text{weakly in } (L^2(Q))^3 \text{ and weakly in } L^2(0, T; (L^2(\partial \Omega))^3) = (L^2(\Sigma))^3,$$

$$\nabla y_\varepsilon \rightharpoonup \nabla y \quad \text{weakly in } (L^2(Q))^9 \text{ and weakly in } (L^2(\Sigma))^9,$$

$$\partial y_\varepsilon / \partial t \rightharpoonup \partial y / \partial t \quad \text{weakly in } (L^2(Q))^3,$$

$$y_\varepsilon(\cdot, t) \rightharpoonup y(\cdot, t) \quad \text{weakly in } (L^2(Q))^3,$$

$$\Delta y_\varepsilon \rightharpoonup \Delta y \quad \text{weakly in } (L^2(Q))^3,$$

$$\nabla p_\varepsilon \rightharpoonup \nabla p \quad \text{weakly in } (L^2(Q))^3.$$

Since $y_\varepsilon \in (W^{1, \infty}(\Omega))^3$, we also have

$$(y_\varepsilon \cdot \nabla) y_\varepsilon + (y_\varepsilon \cdot \nabla) y_\varepsilon \rightharpoonup (y_\varepsilon \cdot \nabla) y + (y \cdot \nabla) y_\varepsilon \quad \text{weakly in } (L^2(Q))^3 \text{ as } \varepsilon \to 0.$$

So, letting $\varepsilon \to 0$ in (4.1), where $u, y$ and $p$ are replaced by $u_\varepsilon, y_\varepsilon$ and $p_\varepsilon$, respectively, we obtain that $u, y$ and $p$ satisfy equations (4.1), too. Moreover, by (4.42) we have

$$\int_\Omega |y(x, T)|^2 dx \leq \liminf_{\varepsilon \to 0} \int_\Omega |y_\varepsilon(x, T)|^2 dx = 0,$$

whence it follows that

$$y(x, T) = 0 \quad \text{a.e. } x \in \Omega.$$

Since $u_\varepsilon$ weakly converges to $u$ in $(L^2(Q))^3$ as $\varepsilon \to 0$, it weakly converges to $u$ in both $(L^2(Q_\omega))^3$ and $(L^2(Q \setminus Q_\omega))^3$, too. By (4.41), the weak convergence of $u_\varepsilon$ in $(L^2(Q_\omega))^3$ leads to the inequality

$$\int_{Q_\omega} e^{-2\delta \beta} |u|^2 dt \leq \liminf_{\varepsilon \to 0} \int_{Q_\omega} e^{-2\delta \beta} |u_\varepsilon|^2 dt$$

$$(4.43) \leq c \left( |y_0|^2_{(L^2(\Omega))^3} + \int_Q e^{-2\delta \beta} \frac{1}{(T-t)^8} |f|^2 dt \right)$$

thanks to the convexity of the functional $u \mapsto \int_Q e^{-2\delta \beta} |u|^2 dt$. The weak convergence in $(L^2(Q \setminus Q_\omega))^3$ assures us that $u = 0$ a.e. in $Q \setminus Q_\omega$ because $u_\varepsilon = 0$ outside $Q_\omega$ for all $\varepsilon > 0$. Thus, $u \in (L^2(Q, e^{-2\delta \beta}))^3$.
Now let us show that $y$ has the same decay at $t = T$. Since the inclusion $(H^{2,1}(Q))^3 \subset L^2(0,T;(H^1(\Omega))^3)$ is compact, we have on a subsequence of $\{\varepsilon\}$ (also denoted by $\{\varepsilon\}$),

$$y_\varepsilon \to y \text{ strongly in } L^2(0,T;(H^1(\Omega))^3) \text{ as } \varepsilon \to 0.$$  

Consequently,

$$y_\varepsilon \to y \text{ a.e. in } Q \text{ as } \varepsilon \to 0.$$  

We also have

$$\hat{\beta}_\varepsilon \to \hat{\beta} \text{ pointwise in } Q \text{ as } \varepsilon \to 0.$$  

So, from (4.41), using Fatou’s lemma,

$$\begin{align*}
\int_Q e^{-2s\delta}\hat{\beta}|y|^2 \, dx \, dt &\leq \liminf_{\varepsilon \to 0} \int_Q e^{-2s\delta}\hat{\beta}_\varepsilon|y_\varepsilon|^2 \, dx \, dt \\
&\leq c \left( |y_0|^2_{(L^2(\Omega))^3} + \int_Q e^{-2s\beta} \frac{1}{(T-t)^{\delta}} |f|^2 \, dx \, dt \right).
\end{align*}$$

(4.44)

It remains to prove only that $e^{-s\delta'}y \in (H^{2,1}(Q))^3$ for $0 < \delta' < \delta$. To this end, we shall pass from $y$ and $p$ to $\tilde{y} = e^{-s\delta'}y$ and $\tilde{p} = e^{-s\delta'}p$. By (4.1), $\tilde{y}$ and $\tilde{p}$ clearly satisfy

$$\begin{align*}
\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} + (y_\varepsilon \cdot \nabla)\tilde{y} + (\tilde{y} \cdot \nabla)y_\varepsilon + \nabla \tilde{p} &= e^{-s\delta'}f + e^{-s\delta'}\chi_{\omega}u - s\delta' e^{-s\delta'}\beta' y \\ 
\text{div } \tilde{y} &= 0 \quad \text{in } Q, \\
\tilde{y} \cdot N &= 0 \quad \text{on } \Sigma, \\
\sum_{i,j=1}^{3} \left( \frac{\partial \tilde{y}_i}{\partial x_j} + \frac{\partial \tilde{y}_j}{\partial x_i} \right) N_i T_j &= 0 \quad \text{on } \Sigma, \\
\tilde{y}(\cdot,0) &= e^{-s\delta'(0)}y_0 \quad \text{in } \Omega.
\end{align*}$$

Using estimate (4.2) for $\tilde{y}$, together with inequalities (4.43) and (4.44), we obtain
because $\delta' < \delta$. Thus, the proof of Theorem 4.1 is finished.

5 Proof of Theorem 2.1

First we transform the problem of controllability of the stationary solution $y_e$ for equations (2.1) into a problem of controllability of the null solution for slightly modified Navier–Stokes equations, with null external forces, by subtracting equations (2.1) and (2.3). The difference $y - y_e$, also denoted by $y$, satisfies

$$
\frac{\partial y}{\partial t} - \Delta y + (y \cdot \nabla)y + (y_e \cdot \nabla)y + (y \cdot \nabla)y_e + \nabla p = \chi_{\omega}u \quad \text{in} \ Q,
$$

$$\text{div} \ y = 0 \quad \text{in} \ Q,$$

$$y \cdot N = 0 \quad \text{on} \ \Sigma,$$

$$\sum_{i,j=1}^{3} \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) N_i T_j = 0 \quad \text{on} \ \Sigma,$$

for all tangential vector fields $T = (T_1, T_2, T_3)$ on $\partial \Omega$,

$$y(\cdot, 0) = y_0 \quad \text{in} \ \Omega.$$

(We have denoted the difference $y_0 - y_e$ by $y_0$, too.) Thus, we have reduced the original controllability problem to that of finding $(u, y, p)$ which satisfies (5.1) but also $y(\cdot, T) = 0$ a.e. in $\Omega$.

Now we shall interpret the null controllability problem for (5.1) as an "invertibility" problem for a certain nonlinear map. To define this map, we need two suitable function spaces.
Let $s > 0$, $\lambda > 0$ and $\delta_1 \in (1/2, 1)$, given by Theorem 4.1, and let $1/2 < \delta' < \delta_1$. We denote by $X(Q)$ the linear space of all $(u, y, p) \in (L^2(Q))^3 \times (H^{2,1}(Q))^3 \times L^2(0,T;H^1(\Omega))$ which satisfy: $e^{-s\delta' \hat{\beta}}y \in (H^{2,1}(Q))^3$, $\partial y/\partial t - \Delta y + (y e \cdot \nabla)y + (y \cdot \nabla)y e + \nabla p - \chi_\omega u \in (L^2(Q,(T-t)^{-8}e^{-2s\beta}))^3$, and
\[
\begin{align*}
\text{div } y &= 0 \quad \text{ in } Q, \\
y \cdot N &= 0 \quad \text{ on } \Sigma, \\
\sum_{i,j=1}^3 \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) N_i T_j &= 0 \quad \text{ on } \Sigma, \\
\text{ for all tangential vector fields } T = (T_1, T_2, T_3) \text{ on } \partial \Omega, \\
y(\cdot, T) &= 0 \quad \text{ a.e. in } \Omega.
\end{align*}
\]

We endow the space $X(Q)$ with the norm
\[
|\!(u, y, p)|_{X(Q)} = \left( |u|^2_{L^2(Q)} + |e^{-s\delta' \hat{\beta}}y|^2_{(H^{2,1}(Q))^3} + |p|^2_{L^2(0,T;H^1(\Omega))} \right) + \int_Q \frac{1}{(T-t)^8} e^{-2s\beta} \left| \frac{\partial y}{\partial t} - \Delta y + (y e \cdot \nabla)y + (y \cdot \nabla)y e + \nabla p - \chi_\omega u \right|^2 dx dt \right)^{1/2}.
\]

In this way $X(Q)$ has become a Banach space. (One also uses Fatou’s lemma to get this.) We define $Y(Q)$ as
\[
Y(Q) = \left( L^2 \left( Q, \frac{1}{(T-t)^8} e^{-2s\beta} \right) \right)^3 \times V,
\]
which becomes a Banach space, too, if we endow it with the usual product norm:
\[
|\!(f, y_0)|_{Y(Q)} = \left( \int_Q \frac{1}{(T-t)^8} e^{-2s\beta} \left| f \right|^2 dx dt + |y_0|^2_V \right)^{1/2}.
\]

Now we define the map $\mathcal{A} : X(Q) \rightarrow Y(Q)$ as follows: for $(u, y, p) \in X(Q),
\[
\mathcal{A}(u, y, p) = \left( \frac{\partial y}{\partial t} - \Delta y + (y e \cdot \nabla)y + (y \cdot \nabla)y e + (y \cdot \nabla)y e + \nabla p - \chi_\omega u, y(\cdot, 0) \right).
\]

Let us show that $\mathcal{A}(u, y, p) \in Y(Q)$ whenever $(u, y, p) \in X(Q)$. It suffices to verify that $(y e \cdot \nabla)y \in (L^2(Q,(T-t)^{-8}e^{-2s\beta}))^3$. Using Schwarz’s inequality
and the inequality expressing the continuity of inclusion $H^1(\Omega) \subset L^4(\Omega)$, we have

$$
\int_Q \frac{1}{(T-t)^8} e^{-2 s \beta} |(y \cdot \nabla)y|^2 \, dx \, dt
$$

$$
\leq c_1 \int_0^T \int_\Omega \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} y \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} \nabla y \right)^2 \, dx \, dt
$$

$$
\leq c_1 \int_0^T \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} y \right)^2 \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} \nabla y \right)^2 \, dt
$$

$$
\leq c_2 \int_0^T \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} y \right)^2 \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} \nabla y \right)^2 \, dt
$$

$$
\leq c_3 \left( \frac{1}{(T-t)^2} e^{-\frac{1}{2} s \beta} y \right)^4 \left( H^2, 1 \right) (Q) \leq c_3 |e^{-s \beta} y|_{(H^2, 1) (Q)}^4 < \infty,
$$

because $1/2 < \delta'$. The map $A$ is also continuously differentiable. Indeed, the differential of $A$ is

$$
(dA)(v, z, q)(u, y, p)
$$

$$
= \left( \frac{\partial y}{\partial t} - \Delta y + (z \cdot \nabla) y + (y \cdot \nabla) z + (y e \cdot \nabla) y + (y e \cdot \nabla) y e + \nabla p - \chi \omega v, \psi, 0 \right),
$$

and one can show (as before) that for some positive constant $c$,

$$
\int_Q \frac{1}{(T-t)^8} e^{-2 s \beta} |(z \cdot \nabla)y + (y \cdot \nabla) z|^2 \, dx \, dt
$$

$$
\leq c |e^{-s \beta} z|_{(H^2, 1) (Q)}^2 |e^{-s \beta} y|_{(H^2, 1) (Q)}^2.
$$

Besides, we have $A(0, 0, 0) = (0, 0)$. So, to prove Theorem 2.1, we have to show that there is $\eta > 0$ such that for any $(f, y_0) \in Y(Q)$ satisfying

$$
|\langle f, y_0 \rangle |_{Y(Q)} < \eta
$$

there exists $(u, y, p) \in X(Q)$ such that

$$
A(u, y, p) = (f, y_0).
$$

(In fact, for our purpose, we can take $f = 0$.) By an infinite-dimensional variant of the implicit function theorem (see [1], p. 101, or [10]), a sufficient
condition assuring such a local invertibility property for $A$ around $(0, 0, 0)$ is that $(dA)(0, 0, 0) : X(Q) \rightarrow Y(Q)$ should be an epimorphism. But this condition can be equivalently expressed as the global null controllability property for the linear equations (4.1), asserted by Theorem 4.1 and which has just been proved. So, the proof of Theorem 2.1 is finished.

References


