Hierarchical Patchy Method for Hybrid Optimal Control

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Outline

• Hybrid Optimal Control and Maximum Principle
• Numerical Approach
• Play Operator Model
• Transversality Conditions and Switching Manifolds
• Three-dimensional Model with One Switch and Two Switches on Time Invariant Submanifolds
• Conclusions and Future Work
Hybrid Optimal Control

Let \([0, \tau_1, \tau_2, \ldots, \tau_N]\) be the sequence of switching times (and associated mode sequences \([k_1, k_2, \ldots, k_m]\)).

\[
\min_u \sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} l_k(x, u_k) \, dt + V(x(T))
\]

subject to

\[
\dot{x} = \begin{cases} 
  f_1(x, u) \\
  f_2(x, u) \\
  \vdots \\
  f_m(x, u) 
\end{cases}
\]

\[x(\tau_0) = x_0\]

Find the optimal switching times as well as the optimal control and cost.
Define a Hamiltonian

\[ H_k(\lambda, x, u) = \lambda^T f_k(x, u_k) - L_k(x, u_k) \]

and the Hamiltonian system: \( \dot{x} = \frac{\partial H_k}{\partial \lambda} \) and \( -\dot{\lambda} = \frac{\partial H_k}{\partial x} \).

**Theorem:** If \( u \) and \( x \) are admissible optimal control and state trajectories of HOC, then

- \((\lambda, x, u)\) satisfies the Hamiltonian system
- for any \((\lambda, x)\) at \( t \), the maximum condition holds:
  \[ H_k(\lambda, x, u) \leq H_k(\lambda, x, v) \]
  for \( v \in \mathcal{U} \), a.e. \( t \in [0, \tau_N] \)
- at autonomous switching times \( \tau_i \), transversality condition holds
  \[ \lambda_j^+ = \lambda_j^- - \frac{\partial m_{j,k}}{\partial x} p|_{t=\tau_i} \quad \text{and} \quad H_j^+ = H_j^- - \frac{\partial m_{j,k}}{\partial t} p|_{t=\tau_i} \]
Numerical Approach: Hierarchical Patchy Technique

Idea: Iterative method for $[0, \tau_1^k, \tau_2^k, \ldots, \tau_n]$ while solving for the optimal control and cost via the patchy technique for HJB at each interval $[\tau_i^k, \tau_j^k]$

- Partition $[0, \tau_1^0, \tau_2^0, \ldots, \tau_{N-1}^0, \tau_N^0]$
- for $k=1:\text{MAX}$
  - Solve HJB on each $[\tau_i^k, \tau_j^k]$
  - $[p_n^k, \tau_n^k] \leftarrow \min \mathcal{G}(x_n^k, \lambda_n^k)$ for all $n$

end
The Patchy Method

- A good local approximation of the HJB PDE
- Look for power series solutions of $\pi(x)$ and $\kappa(x)$
- Substitute power series $f$ and $l$ into HJB and collect same degree terms
- Reduce solving nonlinear PDE to one Riccati and several linear equations

- Use Al'brecht's Method to compute an approximation of the optimal cost containing the origin.
- At a point on the level set which we call *patch point*, compute the partial derivatives of the optimal cost and optimal feedback using the Cauchy-Kovalevskaya technique.
- Use these to approximate the solution on a patch attached to the original patch.
- Repeat this at other points on the level set to get a number of patches that surround the original patch.
- Compute a larger sublevel set and repeat.
Linear Play or Backlash

The rod with position $w(t)$ is moved by a cart of width $2r$ with center position $v(t)$. As long as the rod remains within the interior of the cart, the rod does not move. Once one end of the cart reaches the rod, the rod will move with the cart.
More on Play Operator

- Occurs when several parts that work together (gears) are not perfectly fitted.
- Describes the lost motion due to clearance or slackness when movement is reversed until contact is re-established.
- Behavior limits the performance of speed and position control in robotics, automotive and automation applications.
Standard Play Operator Model

Play operator:

\[ w = \mathcal{P}[v] \]

for any piecewise monotone input function \( v : [0, \tau_N] \to \mathbb{R} \) with

\[
\begin{align*}
    w(0) &= f_r(v(0), 0) \\
    w(t) &= f_r(v(t), w(t_i)), \quad \tau_i < t < \tau_{i+1}
\end{align*}
\]

with

\[ w(t_i) = \max\{v(t) - r, \min\{v(t) + r, w(t_i)\}\} \]

where \( v \) is monotone on each of the subintervals for all \([\tau_i, \tau_{i+1}]\)
Play Operator Model as Hybrid Dynamical System

- The rigid body of width $2r$ with center position $\nu$ moved by a force $u$: $\ddot{\nu} = u(t)$
- Let $x = (\nu, \dot{\nu}, w)$. 

$$\dot{x} = \begin{cases} A_1x + Bu & \text{if } x_1 - x_3 = r \text{ and } x_2 > 0 \\ A_1x + Bu & \text{if } x_1 - x_3 = -r \text{ and } x_2 < 0 \\ A_0x + Bu & \text{otherwise} \end{cases}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
Analysis of the System

Def’n: If there is an absolutely cont. function $x$ defined on a real interval of time $I$ such that $\dot{x} = f(x)$ for all $t \in I$, except possibly on a set of Lebesgue measure zero, then $x$ is said to be a Caratheodory solution of the differential equation.

- The system has a Caratheodory solution $x(\cdot) \in C([0, T]; \mathbb{R}^3)$ for every initial condition $x_0$ and every $u \in C(0, T)$.
- The system of equations has a unique solution $x(t)$ due to the non-diverging property:
  $f$ is non-diverging if $\langle x(t) - y(t), f(x(t), t) - f(y(t), t) \rangle \leq 0$ for all $x, y \in \mathbb{R}^n$ and $t \in [0, \infty)$

Only check for $x_3$ since closed form solutions $x_1(t)$ and $x_2(t)$ can be calculated.
Hybrid Optimal Control

Let \([0, \tau_1, \tau_2, \ldots, \tau_N]\) be the sequence of switching times (and associated mode sequences \([k_1, k_2, \ldots, k_i, \ldots, k_m]\)).

\[
\min_u \sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} \frac{1}{2}(x^T Qx + u^2) \, dt + V(x(\tau_N))
\]

subject to

\[
\dot{x} = \begin{cases} 
A_1x + Bu & \text{if } x_1 - x_3 = r \text{ and } x_2 > 0 \\
A_1x + Bu & \text{if } x_1 - x_3 = -r \text{ and } x_2 < 0 \\
A_0x + Bu & \text{otherwise}
\end{cases}
\]

\[x(\tau_0) = x_0\]

Find the optimal switching times as well as the optimal control and cost.
Current Numerical Methods

- Gradient descent updates and requires the Hamiltonian system solved at iteration (Shaikh and Caines)
- Geodesic-gradient flow algorithm (Taringoo and Caines)
- Convex optimization method (Xu and Antsaklis)
Hybrid Maximum Principle

Define a Hamiltonian

\[ H_k(\lambda, x, u) = \lambda^T f_k(x, u_k) - L_k(x, u_k) \]

and the Hamiltonian system: \( \dot{x} = \frac{\partial H_k}{\partial \lambda} \) and \( -\dot{\lambda} = \frac{\partial H_k}{\partial x} \).

**Theorem:** If \( u \) and \( x \) are an admissible optimal control and state trajectory of HOC, then

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  \[ H_k(\lambda, x, u) \leq H_k(\lambda, x, v) \]
  for \( v \in \mathcal{U} \), a.e. \( t \in [\tau_0, \tau_N] \)
- at autonomous switching times \( \tau_i \), transversality condition holds
  \[ \lambda_j^+ = \lambda_k^- - \frac{\partial m_{j,k}}{\partial x} p|_{t=\tau_i} \quad \text{and} \quad H_j^+ = H_k^- - \frac{\partial m_{j,k}}{\partial t} p|_{t=\tau_i} \]
Switching Submanifolds

Def’n: A time independent submanifold with autonomous transition from mode \( p \) to mode \( q \) occurring at \( \tau_i \) is a smooth codimension one submanifold of \( \mathbb{R}^n \), denoted as \( m_{p,q} \in \mathbb{R}^{n-1} \) described by the zero-level set,

\[
m_{p,q} = \{ x | m_{p,q}(x) = 0 \}
\]

Our Submanifolds:
- \( m_{1,3}(x) = x_1 - x_3 - 1 \)
- \( m_{2,3}(x) = x_2 \)
- \( m_{3,2}(x) = x_1 - x_3 + 1 \)
Nonlinear Least-Squares for HOC

- Minimize $\| g(x) \|^2 = \sum_{i=1}^{m} g_i(x)^2$
  where $g_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, \ldots, m$ are the constraints functions.

- Taylor expanding $g$ at about $x^k = [p^k \ t_s^k]$, we obtain
  \[ g(x) \approx g(x^k) + Dg(x^k)(x - x^k) \]
  where $(Dg)_{ij} = \frac{\partial g_i}{\partial x_j}$

- Let $A^k = Dg(x^k)$. Then,
  \[ 0 \approx A^k x - (A^k x^k - g(x^k)) \approx A^k x - b^k \]
  where $b^k = A^k x^k - g(x^k)$.

- Thus,
  \[ \min \| g(x) \|^2 = \min \| A^k x - b^k \|^2. \]
Constraints for one-switch

Transversality conditions:

\[ g_1(p, t_s) = \lambda_2^+(t_s) - \lambda_2^-(t_s) + p \]
\[ g_2(p, t_s) = (\lambda_2^+(t_s))^2 - (\lambda_2^+(t_s))^2 \]

Switching manifold conditions:

\[ g_3(p, t_s) = x_2^+(t_s) \]
\[ g_4(p, t_s) = x_2^-(t_s) \]

Continuity constraints:

\[ g_5(p, t_s) = x_1^+(t_s) - x_1^-(t_s) \]
\[ g_6(p, t_s) = x_2^+(t_s) - x_2^-(t_s) \]
\[ g_7(p, t_s) = x_3^+(t_s) - x_3^-(t_s) \]

The Jacobian is

\[
Dg = \begin{bmatrix}
1 & \dot{\lambda}_2^+(t_s) - \dot{\lambda}_2^-(t_s) \\
0 & 2\lambda_2^+(t_s)\dot{\lambda}_2^+(t_s) - 2\lambda_2^-(t_s)\dot{\lambda}_2^-(t_s) \\
0 & \dot{x}_2^+(t_s) \\
0 & \dot{x}_2^-(t_s) \\
0 & \dot{x}_1^+(t_s) - \dot{x}_1^-(t_s) \\
0 & \dot{x}_2^+(t_s) - \dot{x}_2^-(t_s) \\
0 & \dot{x}_3^+(t_s) - \dot{x}_3^-(t_s)
\end{bmatrix}
\]

with \((Dg)_{ij} = \frac{\partial g_i}{\partial x_j}\)

Important Note:
• \(x^\pm(t_s)\) and \(\lambda^\pm(t_s)\) obtained from the Hamiltonian system, requiring DRE solutions \(\pi^+\) and \(\pi^-\) on the intervals \([0, \tau_s]\) and \([\tau_s, T]\)
Numerical Results: One Switch

Residual

Transversality Parameter

Switching Time
Numerical Results: Two Switches

Residual

Transversality Parameters

Switching Times
Conclusions and Future Work

• Formulated a hybrid optimal control problem for the optimal switching times of the play operator
• Calculated the optimal solutions of the time varying Riccati (HJB) equations on non-switching time intervals
• Estimating the switching times via the Hybrid Pontryagin Maximum Principle through iterative convex programming

Future Work
- Examples with multi-switches, time dependent switching manifolds
- Improve NLS using alternating directions
- Convergence analysis of the HP method
- Characterization of the optimal solutions in the viscosity sense

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The End

Thank you for your attention!
References

