Improved Patchy Solutions to the Hamilton-Jacobi-Bellman PDE

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The problem to be solved

Find the optimal cost

$$\pi(x^0) \equiv \inf_u \int_0^\infty l(x, u) dt; \quad x \in \mathbb{R}^n \ u \in \mathbb{R}^m$$

subject to the dynamics

$$\dot{x} = f(x, u) \quad \quad x(0) = x^0$$

$$l(x, u) \geq 0 \quad l(x, u) = 0 \text{ only if } x = 0 \text{ and } u = 0$$

$u$: control

$\pi$: optimal cost

The control that yields the optimal cost is called the optimal control
Hamilton-Jacobi-Bellman PDEs

If

- The optimal cost $\pi$ is smooth
- The optimal control is a smooth function of the state $u^{\text{opt}} = \kappa(x)$
- $\frac{\partial \pi}{\partial x}(x)f(x, u) + l(x, u)$ is strictly convex in $u$ for every $x$ in a region containing $x = 0, u = 0$

Then the solutions to the Hamilton-Jacobi-Bellman PDEs are the optimal cost and optimal control

\[
0 = \frac{\partial \pi}{\partial x}(x)f(x, \kappa(x)) + l(x, \kappa(x))
\]

\[
0 = \frac{\partial \pi}{\partial x}(x)\frac{\partial f}{\partial u}(x, \kappa(x)) + \frac{\partial l}{\partial u}(x, \kappa(x))
\]
Al’brekht’s method

Al’brekht’s method (1961) computes a series solution to the optimal control problem centered at the origin of the state space

\[ f(x, u) = Fx + Gu + f^{[2]}(x, u) + f^{[3]}(x, u) + \cdots \]

\[ l(x, u) = \frac{1}{2} (x^T Q x + u^T R u) + l^{[3]}(x, u) + l^{[4]}(x, u) + \cdots \]

Standard linear-quadratic regulator problem assumptions on \( F, G, Q, \) and \( R \):

- \( Q \succeq 0 \quad R \succ 0 \)
- \( (F, G) \) stabilizable: All eigenvalues of \( F + GK \) have strictly negative real parts for some \( K \)
- \( (F, Q^{1/2}) \) detectable: All eigenvalues of \( F + LQ^{1/2} \) have strictly negative real parts for some \( L \)
Al’brekht’s method

Albrekht’s method computes the Taylor expansions of $\pi$ and $\kappa$

$$\pi(x) = \frac{1}{2}x^T P x + \pi^{[3]}(x) + \pi^{[4]}(x) + \cdots$$

$$\kappa(x) = K x + \kappa^{[2]}(x) + \kappa^{[3]}(x) + \cdots$$

- Plug expansions of $f$, $l$, $\pi$, $\kappa$ into the HJB PDEs
- Group terms by powers of $x$ and set coefficients to zero
- Solve a sequence of equation pairs
Why extend Al’brekht’s method?

- Series solution generated by Al’brekht’s method must be centered at $x = 0$
- Increasing number of terms in series solution may not increase accuracy outside some neighborhood of $x = 0$
- Patchy algorithm can expand the domain of validity
Patchy algorithm  Navasca and Krener 2007, revision by Hunt and Krener 2011

The patchy algorithm grows the solution to the HJB PDEs out from the origin

Initialization:

- Start with series solutions $\pi^0(x)$ and $\kappa^0(x)$ generated by Al’brekht’s method
- Set Al’brekht patch boundary as level set of $\pi^0$ such that HJB PDEs are nearly satisfied on the sublevel set
Patchy algorithm

Main loop:
▶ Pick patch point $x^1$ on boundary of existing Al’brekht patch
▶ Compute approximating polynomials $\pi^1(x)$ and $\kappa^1(x)$ to optimal cost and optimal control
▶ Pick lateral patch boundaries of newly created patch

Repeat until the Al’brekht patch is surrounded by new patches
Enclose ring of new patches with a level set of the optimal cost
Patchy algorithm

Al’brekht patch $A$

$x^0 = 0$

$\pi^1(x)$

$\kappa^1(x)$

$\pi^0(x)$

$\kappa^0(x)$

$\pi^0(x) = \text{constant}$
Patchy algorithm

$\pi^0(x) = \text{constant}$

$\pi^1(x)$

$\kappa^1(x)$

$x^0 = 0$

$\text{Al’brekht patch } A$

$\pi^0(x) = \text{constant}$

$\pi^1(x)$

$\kappa^1(x)$
Patchy algorithm

\[ x^0 = 0 \]

\[ x^1 \]

\[ \pi^1(x) \]
\[ \kappa^1(x) \]

\[ \pi^0(x) \]
\[ \kappa^0(x) \]

\[ \text{Al’brekht patch } A \]

\[ \pi^0(x) = \text{constant} \]
Patchy algorithm

Al’brekht patch $A$

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Al’brekht patch $A$

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Patchy algorithm

\[ x^0 = 0 \]

\[ x^1 \]

\[ \pi^1(x) \]

\[ \kappa^1(x) \]

Al’brekht patch \( A \)

\[ \pi^0(x) = \text{constant} \]
Patchy algorithm

\[ x^0 = 0 \]

\[ x^1 \]

\[ \pi^1(x) \]
\[ \kappa^1(x) \]

\[ \pi^0(x) \]
\[ \kappa^0(x) \]

Al’brekht patch \( A \)

\[ \pi^0(x) = \text{constant} \]
Patchy algorithm assumptions

We assume the problem data $f$ and $l$ have the form

\[
f(x, u) = f(x) + g(x)u \quad l(x, u) = q(x) + \frac{1}{2} r(x) u^2
\]

$r(x) > 0$ everywhere, $q(x) > 0$ away from $x = 0$
Patchy algorithm assumptions

We assume the problem data \( f \) and \( l \) have the form

\[
\begin{align*}
  f(x, u) &= f(x) + g(x)u \\
  l(x, u) &= q(x) + \frac{1}{2} r(x)u^2
\end{align*}
\]

\( r(x) > 0 \) everywhere, \( q(x) > 0 \) away from \( x = 0 \)

- Ensures \( \frac{\partial \pi}{\partial x}(x)f(x, u) + l(x, u) \) is strictly convex in \( u \) for every \( x \)
- For simplicity: \( x \in \mathbb{R}^2 \) and \( u \in \mathbb{R} \)

Hamilton-Jacobi-Bellman equations are

\[
\begin{align*}
  0 &= \frac{\partial \pi}{\partial x}(x)(f(x) + g(x)\kappa(x)) + q(x) + \frac{1}{2} r(x)\kappa(x)^2 \\
  0 &= \frac{\partial \pi}{\partial x}(x)g(x) + r(x)\kappa(x)
\end{align*}
\]
Patchy algorithm assumptions

The exact optimal cost $\pi$ is a strict Lyapunov function on the computational domain $X_c$

- $\pi(x) > 0$ on $X_c \setminus \{0\}$ and $\pi(0) = 0$
- $\frac{\partial \pi}{\partial x}(x) \dot{x} < 0$ on $X_c \setminus \{0\}$
The exact optimal cost $\pi$ is a strict Lyapunov function on the computational domain $X_c$

- $\pi(x) > 0$ on $X_c \setminus \{0\}$ and $\pi(0) = 0$
- $\frac{\partial \pi}{\partial x}(x) \dot{x} < 0$ on $X_c \setminus \{0\}$

Ensures that

- $\dot{x} \neq 0$
- $\frac{\partial \pi}{\partial x}(x) \neq 0$

on $X_c \setminus \{0\}$
Computing $\pi^1$ and $\kappa^1$

- $\pi^1$ and $\kappa^1$ are polynomials centered at $x^1$
- Ideally, we would compute the Taylor expansion of $\pi$ and $\kappa$ centered at $x^1$
- Compute approximate mixed partial derivatives of $\pi$ and $\kappa$ at $x^1$
Old Patchy: Computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

Pick *characteristic index* $k$ that maximizes $|f_k(x^1) + g_k(x^1)\kappa^0(x^1)|$
(Assume $k = n$ for simplicity)

Calculate non-characteristic partial derivatives by inheritance

$$\frac{\partial \pi^1}{\partial x_i}(x^1) = \frac{\partial \pi^0}{\partial x_i}(x^1) \text{ for } 1 \leq i < n$$
Pick *characteristic index* $k$ that maximizes $|f_k(x^1) + g_k(x^1)\kappa^0(x^1)|$
(Assume $k = n$ for simplicity)
Calculate non-characteristic partial derivatives by inheritance

$$\frac{\partial \pi^1}{\partial x_i}(x^1) = \frac{\partial \pi^0}{\partial x_i}(x^1) \text{ for } 1 \leq i < n$$

Solve second HJB equation for $\kappa(x)$ in terms of $\frac{\partial \pi}{\partial x}(x)$ and substitute into first HJB equation

$$0 = -\frac{1}{2r(x)} \left( \frac{\partial \pi}{\partial x}(x) \cdot g(x) \right)^2 + \frac{\partial \pi}{\partial x}(x) \cdot f(x) + q(x)$$

Substitute the non-characteristic partial derivatives into the HJB equation and solve for the characteristic partial derivative $\frac{\partial \pi^1}{\partial x_n}(x^1)$
Old Patchy: Computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

The characteristic partial derivative $\frac{\partial \pi^1}{\partial x_n}(x^1)$ is the solution of the quadratic

$$0 = a \left( \frac{\partial \pi^1}{\partial x_n}(x^1) \right)^2 + b \frac{\partial \pi^1}{\partial x_n}(x^1) + c$$

closest to $\frac{\partial \pi^0}{\partial x_n}(x^1)$, where

$$a = \frac{1}{2r(x^1)}(g_n(x^1))^2$$

$$b = \frac{1}{2r(x^1)} g_n(x^1) \sum_{\sigma=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) g_\sigma(x) - f_n(x^1)$$

$$c = \frac{1}{2r(x^1)} \sum_{\sigma=1}^{n-1} \sum_{\tau=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) g_\sigma(x^1) \frac{\partial \pi^1}{\partial x_\tau}(x^1) g_\tau(x^1)$$

$$- \sum_{\sigma=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) f_\sigma(x^1) - q(x^1)$$
Old Patchy: Computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

The computed optimal $\kappa^1(x^1)$ control is computed from $\frac{\partial \pi^1}{\partial x}(x^1)$

\[ \kappa^1(x^1) = -\frac{1}{r(x^1)} \frac{\partial \pi^1}{\partial x}(x^1) \cdot g(x^1) \]
Old Patchy: Computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

$$0 = -\frac{1}{2r(x^1)}(v \cdot g(x^1))^2 + v \cdot f(x^1) + q(x^1)$$
The HJB equations do not specify all the partial derivatives of \( \pi \) away from the origin.
Old Patchy Computing \(\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)\) and \(\frac{\partial \kappa^1}{\partial x_i}(x^1)\)

- The HJB equations do not specify all the partial derivatives of \(\pi\) away from the origin
- We compute the second order and higher partial derivatives of \(\pi\) by two methods
Old Patchy: Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

Compute second order non-characteristic partial derivatives by inheritance

$$\frac{\partial \pi^1}{\partial x_i \partial x_j}(x^1) = \frac{\partial \pi^0}{\partial x_i \partial x_j}(x^1) \text{ for } 1 \leq i \leq j \leq n - 1$$
Old Patchy: Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

Compute second order non-characteristic partial derivatives by inheritance

$$\frac{\partial \pi^1}{\partial x_i \partial x_j}(x^1) = \frac{\partial \pi^0}{\partial x_i \partial x_j}(x^1) \text{ for } 1 \leq i \leq j \leq n - 1$$

Take partial derivative of first HJB equation with respect to $x_i$

$$\frac{\partial^2 \pi}{\partial x_i \partial x}(x) \cdot (f(x) + g(x)\kappa(x)) = -\frac{\partial \pi}{\partial x}(x) \cdot \left( \frac{\partial f}{\partial x_i}(x) + \frac{\partial g}{\partial x_i}(x)\kappa(x) \right)$$

$$- \frac{\partial q}{\partial x_i}(x) - \frac{1}{2} \frac{\partial r}{\partial x_i}(x)(\kappa(x))^2$$

Compute second order characteristic partial derivatives so the derivative HJB equation is satisfied
Old Patchy: Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

Second order characteristic partial derivatives

For $i = 1, \ldots, n$ set characteristic second order partial partial derivatives to

$$\frac{\partial^2 \pi^1}{\partial x_i \partial x_n}(x^1) = -\frac{1}{f_n(x^1) + g_n(x^1)\kappa^1(x^1)} \times$$

$$\left[ \sum_{\sigma=1}^{n-1} \frac{\partial^2 \pi^1}{\partial x_i \partial x_\sigma}(x^1)(f_\sigma(x^1) + g_\sigma(x^1)\kappa^1(x^1)) \right]$$

$$+ \frac{\partial \pi^1}{\partial x}(x^1) \left( \frac{\partial f}{\partial x_i}(x^1) + \frac{\partial g}{\partial x_i}(x^1)\kappa^1(x^1) \right)$$

$$+ \frac{\partial q}{\partial x_i}(x^1) + \frac{1}{2} \frac{\partial r}{\partial x_i}(x^1)(\kappa^1(x^1))^2 \right]$$
Old Patchy: Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

$\frac{\partial \kappa^1}{\partial x}(x^1)$ is computed from $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Take derivative of second HJB equation with respect to $x_i$

$$r(x) \frac{\partial \kappa}{\partial x_i}(x) = - \frac{\partial^2 \pi}{\partial x_i \partial x}(x) \cdot g(x) - \frac{\partial \pi}{\partial x}(x) \cdot \frac{\partial g}{\partial x_i}(x)$$

$$- \frac{\partial r}{\partial x_i}(x) \kappa(x) - \frac{\partial \kappa}{\partial x_i}(x) r(x)$$
Old Patchy: Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

$\frac{\partial \kappa^1}{\partial x}(x^1)$ is computed from $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Take derivative of second HJB equation with respect to $x_i$

$$r(x) \frac{\partial \kappa}{\partial x_i}(x) = -\frac{\partial^2 \pi}{\partial x_i \partial x}(x) \cdot g(x) - \frac{\partial \pi}{\partial x}(x) \cdot \frac{\partial g}{\partial x_i}(x)$$

$$- \frac{\partial r}{\partial x}(x) \kappa(x) - \frac{\partial \kappa}{\partial x_i}(x) r(x)$$

Set

$$\frac{\partial \kappa^1}{\partial x_i}(x^1) = -\frac{1}{r(x^1)} \left[ \frac{\partial^2 \pi^1}{\partial x_i \partial x}(x^1) \cdot g(x^1) + \frac{\partial \pi^1}{\partial x}(x^1) \cdot \frac{\partial g}{\partial x_i}(x^1) + \frac{\partial r}{\partial x_i}(x^1) \kappa(x^1) \right]$$

Higher order partial derivatives of $\pi$ and $\kappa$ are computed analogously
New Patchy Method motivation

Why nudge $\frac{\partial \pi^0}{\partial x}(x^1)$ to $\frac{\partial \pi^0}{\partial x}(x^1)$ on solution curve along a coordinate axis?

$$\frac{\partial \pi^0}{\partial x}(x^1) = \left( \frac{\partial \pi^0}{\partial x_1}(x^1), \frac{\partial \pi^0}{\partial x_2}(x^1) \right)$$

$$0 = -\frac{1}{2r(x^1)}(v \cdot g(x^1))^2 + v \cdot f(x^1) + q(x^1)$$
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

Inherit the gradient direction of $\pi$ from the previous patch point

$$\mathbf{n} \equiv \frac{1}{\| \frac{\partial \pi^0}{\partial x}(x^1) \|} \frac{\partial \pi^0}{\partial x}(x^1)$$
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

Inherit the gradient direction of $\pi$ from the previous patch point

$$n \equiv \frac{1}{\| \frac{\partial \pi^0}{\partial x}(x^1) \|} \frac{\partial \pi^0}{\partial x}(x^1)$$

Calculate the gradient length $z$ so

$$\frac{\partial \pi^1}{\partial x}(x^1) = zn$$

satisfies the HJB equation
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

\[ n = \frac{\partial \pi^0}{\partial x}(x^1) / \| \frac{\partial \pi^0}{\partial x}(x^1) \| \]

$\frac{\partial \pi^1}{\partial x}(x^1) = zn$
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

Denote the dot product of $f$ and $g$ with the normalized gradient of optimal cost at $x^1$ as

$$f_n \equiv \mathbf{n} \cdot f(x^1) \quad \quad g_n \equiv \mathbf{n} \cdot g(x^1)$$
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

Denote the dot product of $f$ and $g$ with the normalized gradient of optimal cost at $x^1$ as

$$f_n \equiv \mathbf{n} \cdot f(x^1) \quad \quad g_n \equiv \mathbf{n} \cdot g(x^1)$$

The HJB equations reduce to scalar equations

$$0 = -\frac{g_n^2}{2r(x^1)} z^2 + f_n z + q(x^1)$$

$$u = -\frac{g_n}{r(x^1)} z$$

Under the strict Lyapunov assumption, the quadratic has exactly one positive solution $z_+$ if $\frac{\partial \pi^0}{\partial x}(x^1)$ is close enough to $\frac{\partial \pi}{\partial x}(x^1)$
New Patchy: computing $\frac{\partial \pi^1}{\partial x}(x^1)$ and $\kappa^1(x^1)$

We set

$$\frac{\partial \pi^1}{\partial x}(x^1) = z_+ \mathbf{n}$$
We set

$$\frac{\partial \pi^1}{\partial x}(x^1) = z^+ n$$

and

$$\kappa^1(x^1) = -\frac{1}{r(x^1)} \frac{\partial \pi^1}{\partial x}(x^1) g(x^1)$$
New Patchy Method motivation

The derivative of the HJB equation

\[
\frac{\partial^2 \pi}{\partial x_i \partial x}(x) \cdot (f(x) + g(x)\kappa(x)) = -\frac{\partial \pi}{\partial x}(x) \cdot \left( \frac{\partial f}{\partial x_i}(x) + \frac{\partial g}{\partial x_i}(x)\kappa(x) \right) \\
- \frac{\partial q}{\partial x_i}(x) - \frac{1}{2} \frac{\partial r}{\partial x_i}(x) (\kappa(x))^2
\]

only specifies the directional derivative of \( \frac{\partial^2 \pi}{\partial x_i \partial x}(x) \) in the direction of \( \dot{x} = f(x) + g(x)\kappa(x) \), so why not rotate the coordinate system so one coordinate axis is aligned with \( \dot{x} \)?
New Patchy Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Compute higher order partial derivatives of $\pi$ in new variables and then convert back to original variables

$$\nu \equiv \|f(x^1) + g(x^1)\kappa^1(x^1)\|$$

$$\nu^1 \equiv \frac{1}{\nu} \left( f(x^1) + g(x^1)\kappa^1(x^1) \right)$$

$\nu$ is nonzero under the strict Lyapunov assumption
New Patchy Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Compute higher order partial derivatives of $\pi$ in new variables and then convert back to original variables

$$\nu \equiv \|f(x^1) + g(x^1)\kappa^1(x^1)\|$$

$$\nu^1 \equiv \frac{1}{\nu}(f(x^1) + g(x^1)\kappa^1(x^1))$$

$\nu$ is nonzero under the strict Lyapunov assumption

Generate $\nu^2, \nu^3, \ldots, \nu^n$ so

$$V = \begin{bmatrix} \nu^1 & \nu^2 & \cdots & \nu^n \end{bmatrix}$$

is an orthonormal basis (Householder reflector)
Compute higher order partial derivatives of $\pi$ in new variables and then convert back to original variables

\[
\nu \equiv \| f(x^1) + g(x^1) \kappa^1(x^1) \|
\]

\[
\nu^1 \equiv \frac{1}{\nu} (f(x^1) + g(x^1) \kappa^1(x^1))
\]

\(\nu\) is nonzero under the strict Lyapunov assumption

Generate \(\nu^2, \nu^3, \ldots, \nu^n\) so

\[
V = \begin{bmatrix} \nu^1 & \nu^2 & \ldots & \nu^n \end{bmatrix}
\]

is an orthonormal basis (Householder reflector)

New state space variable: \(\xi\)

\[
x = x^1 + V \xi
\]
New Patchy Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

The HJB equations in the new variables are

$$0 = \frac{\partial \tilde{\pi}}{\partial \xi}(\tilde{f}(\xi) + \tilde{g}(\xi)\tilde{\kappa}(\xi)) + \tilde{q}(\xi) + \frac{1}{2} \tilde{r}(\xi)\tilde{\kappa}(\xi)^2$$

$$0 = \frac{\partial \tilde{\pi}}{\partial \xi}(\xi)\tilde{g}(\xi) + \tilde{r}(\xi)\tilde{\kappa}(\xi)$$

where

$$\tilde{\pi}(\xi) \equiv \pi(x^1 + V\xi) \quad \tilde{\kappa}(\xi) \equiv \kappa(x^1 + V\xi)$$

$$\tilde{f}(\xi) \equiv V^T f(x^1 + V\xi) \quad \tilde{g}(\xi) \equiv V^T g(x^1 + V\xi)$$

$$\tilde{q}(\xi) \equiv q(x^1 + V\xi) \quad \tilde{r}(\xi) \equiv r(x^1 + V\xi)$$
New Patchy Computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j} (x^1)$

New coordinate system:

$$x = x^1 + [v^1 \ v^2] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Optimal trajectory:

$$\dot{x} = f(x^1) + g(x^1)\kappa(x^1)$$
New Patchy: computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Differentiate the first HJB equation with respect to $\xi_i$ and evaluate at $\xi = 0$: 

$$0 = \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi}(0)(\tilde{f}(0) + \tilde{g}(0)\tilde{\kappa}(0))$$

$$+ \frac{\partial \tilde{\pi}}{\partial \xi}(0)\left( \frac{\partial \tilde{f}}{\partial \xi_i}(0) + \frac{\partial \tilde{g}}{\partial \xi_i}(0)\tilde{\kappa}(0) \right) + \frac{\partial \tilde{q}}{\partial \xi_i}(0) + \frac{1}{2} \frac{\partial \tilde{r}}{\partial \xi_i}(0)\tilde{\kappa}(0)^2$$
New Patchy: computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

Differentiate the first HJB equation with respect to $\xi_i$ and evaluate at $\xi = 0$:

$$0 = \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi}(0)(\tilde{f}(0) + \tilde{g}(0)\tilde{\kappa}(0))$$

$$+ \frac{\partial \tilde{\pi}}{\partial \xi}(0)\left(\frac{\partial \tilde{f}}{\partial \xi_i}(0) + \frac{\partial \tilde{g}}{\partial \xi_i}(0)\tilde{\kappa}(0)\right) + \frac{\partial \tilde{q}}{\partial \xi_i}(0) + \frac{1}{2} \frac{\partial \tilde{r}}{\partial \xi_i}(0)\tilde{\kappa}(0)^2$$

which reduces to

$$0 = \nu \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi_1}(0)$$

$$+ \frac{\partial \tilde{\pi}}{\partial \xi}(0)\left(\frac{\partial \tilde{f}}{\partial \xi_i}(0) + \frac{\partial \tilde{g}}{\partial \xi_i}(0)\tilde{\kappa}(0)\right) + \frac{\partial \tilde{q}}{\partial \xi_i}(0) + \frac{1}{2} \frac{\partial \tilde{r}}{\partial \xi_i}(0)\tilde{\kappa}(0)^2$$
New Patchy: computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

For $1 \leq i \leq n$ set the characteristic partials to:

$$\frac{\partial^2 \tilde{\pi}^1}{\partial \xi_i \partial \xi_1}(0) = -\frac{1}{\nu^1} \frac{\partial \tilde{\pi}^1}{\partial \xi}(0) \left( \frac{\partial \tilde{f}}{\partial \xi_i}(0) + \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) \right)$$

$$+ \frac{\partial \tilde{q}}{\partial \xi_i}(0) + \frac{1}{2} \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}^1(0)^2$$
New Patchy: computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$

For $1 \leq i \leq n$ set the characteristic partials to:

$$\frac{\partial^2 \tilde{\pi}^1}{\partial \xi_i \partial \xi_1}(0) = -\frac{1}{\nu^1} \frac{\partial \tilde{\pi}^1}{\partial \xi}(0) \left( \frac{\partial \tilde{f}}{\partial \xi_i}(0) + \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) \right)$$

$$+ \frac{\partial \tilde{q}}{\partial \xi_i}(0) + \frac{1}{2} \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}^1(0)^2$$

For $2 \leq i \leq j \leq n$ set the noncharacteristic partials to:

$$\frac{\partial^2 \tilde{\pi}^1}{\partial \xi_i \partial \xi_j}(0) = \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left[ \pi^0(x^1 + V \xi) \right]_{\xi=0}$$
Differentiate the second HJB equation with respect to $\xi_i$ and evaluate at $\xi = 0$:

$$0 = \tilde{r}(0) \frac{\partial \tilde{\kappa}}{\partial \xi_i}(0) + \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}(0) + \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi}(0) \tilde{g}(0) + \frac{\partial \tilde{\pi}}{\partial \xi}(0) \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}(0)$$
Differentiate the second HJB equation with respect to $\xi_i$ and evaluate at $\xi = 0$:

$$0 = \tilde{r}(0) \frac{\partial \tilde{\kappa}}{\partial \xi_i}(0) + \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}(0) + \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi}(0) \tilde{g}(0) + \frac{\partial \tilde{\pi}}{\partial \xi}(0) \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}(0)$$

Set:

$$\frac{\partial \tilde{\kappa}^1}{\partial \xi_i}(0) = -\frac{1}{\tilde{r}(0)} \left( \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) + \frac{\partial^2 \tilde{\pi}^1}{\partial \xi_i \partial \xi}(0) \tilde{g}(0) + \frac{\partial \tilde{\pi}^1}{\partial \xi}(0) \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) \right)$$
New Patchy: computing $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

Differentiate the second HJB equation with respect to $\xi_i$ and evaluate at $\xi = 0$:

$$0 = \tilde{r}(0) \frac{\partial \tilde{\kappa}}{\partial \xi_i}(0) + \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}(0) + \frac{\partial^2 \tilde{\pi}}{\partial \xi_i \partial \xi}(0) \tilde{g}(0) + \frac{\partial \tilde{\pi}}{\partial \xi}(0) \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}(0)$$

Set:

$$\frac{\partial \tilde{\kappa}^1}{\partial \xi_i}(0) = - \frac{1}{\tilde{r}(0)} \left( \frac{\partial \tilde{r}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) + \frac{\partial^2 \tilde{\pi}^1}{\partial \xi_i \partial \xi}(0) \tilde{g}(0) + \frac{\partial \tilde{\pi}^1}{\partial \xi}(0) \frac{\partial \tilde{g}}{\partial \xi_i}(0) \tilde{\kappa}^1(0) \right)$$

The higher order partial derivatives of $\tilde{\pi}^1$ and $\tilde{\kappa}^1$ at $\xi = 0$ are computed analogously.
New Patchy: computing $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$

In $\mathbb{R}^2$ the second order partials of $\pi^1$ with respect to the original state variables are recovered by:

$$
\begin{bmatrix}
\frac{\partial^2 \pi^1}{\partial x_1^2}(x^1) & \frac{\partial^2 \pi^1}{\partial x_1 \partial x_2}(x^1) & \frac{\partial^2 \pi^1}{\partial x_1 \partial x_2}(x^1) & \frac{\partial^2 \pi^1}{\partial x_2^2}(x^1) \\
\frac{\partial^2 \tilde{\pi}^1}{\partial \xi_1^2}(0) & \frac{\partial^2 \tilde{\pi}^1}{\partial \xi_1 \partial \xi_2}(0) & \frac{\partial^2 \tilde{\pi}^1}{\partial \xi_1 \partial \xi_2}(0) & \frac{\partial^2 \tilde{\pi}^1}{\partial \xi_2^2}(0)
\end{bmatrix}
\begin{bmatrix}
V^T \\
V^T
\end{bmatrix}
$$

where $\otimes$ denotes the Kronecker product:

$$
A \otimes B \equiv
\begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix}
$$

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A test problem

Find

$$\pi(x^0) = \inf_u \frac{1}{2} \int_0^\infty \left( \sin^2(x_1) + \left(x_2 - \frac{1}{3} x_1^3\right)^2 + u^2 \right) dt$$

subject to the dynamics

$$\dot{x}_1 = \left(x_2 - \frac{1}{3} x_1^3\right) \sec(x_1)$$

$$\dot{x}_2 = \left(x_1^2 x_2 - \frac{1}{3} x_1^5\right) \sec(x_1) + u$$

Singularity at $x_1 = \frac{\pi}{2} + k\pi!$
A test problem

The test problem is a transformed linear-quadratic regulator problem with a change of coordinates

$$\pi(y^0) = \inf_u \frac{1}{2} \int_0^\infty (y^T y + u^2) \, dt$$

subject to the dynamics

$$\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= u
\end{align*}$$

under the change of coordinates $y_1 = \sin(x_1)$ and $y_2 = x_2 - x_1^3/3$.

$$\pi(y) = \frac{1}{2} y^T \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} y \quad \kappa(y) = - \begin{bmatrix} 1 & \sqrt{3} \end{bmatrix} y$$
New Patchy Optimal Cost $\pi^i(x)$
True Optimal Cost $\pi(x)$
New Patchy Absolute Error: $\text{err}^{\pi}_{\text{abs}} = \pi(x) - \pi^i(x)$

$$|\text{err}^{\pi}_{\text{abs}}| \leq 6.5 \times 10^{-3}$$
Empirical error comparison

<table>
<thead>
<tr>
<th></th>
<th>LQR</th>
<th>Al’brekht</th>
<th>Old Patchy</th>
<th>New Patchy</th>
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<tr>
<td>max absolute error</td>
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<td>.1636</td>
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Absolute and Relative Error
New Patchy method error bound

If

▶ \( \pi \) is a smooth strict Lyapunov function on the computational domain \( X_c \)

Then there exists

▶ maximum consecutive patch point distance \( h \)
▶ Lipschitz constant \( L \) such that

\[
|\pi(x) - \pi_i(x)| \leq C\left(\frac{L^i - 1}{L - 1} + 1\right)h^{d+2} \text{ for } 1 \leq i \leq N
\]

(Superscript \( i \) on \( \pi^i \) is an index and superscript \( i \) on \( L^i \) is a power!)

\( d + 1 \): Degree of approximate optimal cost \( \pi^i \)

\( i \): Number of concentric patch rings between \( x^i \) and \( x^0 \)

\( C \): Prettifying constant

Argument is modeled after error bound for one step methods for solving ODEs
Patchy method error bound

Sequence of patch points $x^0, x^1, \ldots$ and approximating polynomials $\pi^0, \pi^1, \ldots$
Improved Patchy method error bound

It is almost certainly true that there is a better error bound

$$|\pi(x) - \pi^i(x)| \leq C \frac{e^{iLh} - 1}{L} h^{d+1} \text{ for } 1 \leq i \leq N$$

Exponent $iLh$ is proportional to the length of the optimal trajectory starting at $x^i$ and terminating at $x^0 = 0$
Thanks!
Questions?